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Finite iterative algorithm for solving a class of complex matrix equation with two unknowns of general form

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Abstract: This paper is concerned with an efficient iterative algorithm to solve general the Sylvester-conjugate matrix equation of the form $\sum_{i=1}^{s} A_i V B_i + \sum_{j=1}^{t} C_j W D_j = \sum_{l=1}^{m} E_l \overline{V} F_l + C$ The proposed algorithm is an extension to our proposed general Sylvester-conjugate equation of the form $\sum_{i=1}^{s} A_i V + \sum_{j=1}^{t} B_j W = \sum_{l=1}^{m} E_l \overline{V} F_l + C$ When a solution exists for this matrix equation, for any initial matrices, the solutions can be obtained within finite iterative steps in the absence of round off errors. Some lemmas and theorems are stated and proved where the iterative solutions are obtained. Finally, a numerical example is given to verify the effectiveness of the proposed algorithm.

Keywords: General Sylvester-Conjugate matrix Equations, Finite Iterative Algorithm, Orthogonality, Inner Product Space, Frobenius norm

1. Introduction

We know that matrix equation is one of the topics of very active research in computational mathematics, and a large number of papers have presented several methods for solving several matrix equations [1–5]. Ding and Chen presented the hierarchical gradient iterative algorithms for general matrix equations [6, 12] and hierarchical least squares iterative algorithms for generalized coupled Sylvester matrix equations and general coupled matrix equations [13, 14]. The hierarchical gradient iterative algorithms [6, 12] and hierarchical least squares iterative algorithms [6,14,15] for solving general (coupled) matrix equations are innovational and computationally efficient numerical ones and were proposed based on the hierarchical identification principle [13,16] which regards the unknown matrix as the system parameter matrix to be identified.

In [7], the necessary and sufficient conditions for the solvability of the matrix equation $A^H XB = C$, over reflexive and anti-reflexive matrices are given, and the general expression of the reflexive and anti-reflexive solutions for a solvable case is obtained. Ramadan et al. [8] introduced a complete, general and explicit solution to the Yakubovich matrix equation $V - AVF = BW$, with F in an arbitrary form. Also with the help of the concept of Kronecker map, an explicit solution for the matrix equation $XF - AX = C$ was established in [9]. Zhou et al. [10] proposed gradient based iterative algorithms for solving the general coupled Sylvester matrix equations with weighted least squares solutions. In [11], a general parametric solution to a family of generalized Sylvester matrix equations arising in linear system theory is presented by using the so-called generalized Sylvester mapping which has some elegant properties.

In [17], a finite iterative algorithm for solving the generalized $(P,Q)-$reflexive solution of the linear systems of matrix equations was given. Solutions to the so-called coupled Sylvester-conjugate matrix equations, which include the generalized Sylvester matrix equation and coupled Lyapunov matrix equation as special cases are given and presented by A.G. Wu at el. [18]. In [19], an iterative algorithm is presented for solving the extended Sylvester-conjugate matrix equation. Ramadan et. al. proposed a finite iterative solution to general Sylvester-conjugate matrix equation of the form $\sum_{i=1}^{s} A_i V + \sum_{j=1}^{t} B_j W = \sum_{l=1}^{m} E_l \overline{V} F_l + C$ in [20].

This paper is organized as follows: First, in section 2, we introduce some notations, lemmas and theorems that will be needed to develop this work. In section 3, we propose iterative method to obtain numerical solution to the matrix equations $\sum_{i=1}^{s} A_i V + \sum_{j=1}^{t} B_j W = \sum_{l=1}^{m} E_l \overline{V} F_l + C$ using iterative method. In section 4, a numerical example is given to explore the simplicity and the neatness of the presented methods.

2. Preliminaries

The following notations, definitions, lemmas and theorems will be used to develop the proposed work. We use $A^T, \overline{A}, A^H$ and $tr(A)$ to denote the transpose, conjugate, conjugate transpose and the trace of a matrix A respectively. We denote the set of all $m \times n$ complex

matrices by $\mathbb{C}^{m\times n}$, $\mathrm{Re}(a)$ denote the real part of number a.

Definition 1. Inner product [38]

A real inner product space is a vector space V over the real field $\mathbb{R}$ together with an inner product. i.e. with a map

$$\langle .,. \rangle : V \times V \to \mathbb{R}$$

Satisfying the following three axioms for all vectors $x, y, z \in V$ and all scalars $a \in \mathbb{R}$

1. Symmetry: $\langle x, y \rangle = \langle y, x \rangle$.
2. Linearity in the first argument:

$$\langle ax, y \rangle = a \langle x, y \rangle,$$

$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle,$$

3. Positive definiteness: $\langle x, x \rangle > 0$ for all $x \neq 0$.

Two vectors $u, v \in V$ are said to be orthogonal if $\langle u, v \rangle = 0$.

The following theorem defines a real inner product on space $\mathbb{C}^{m\times n}$ over the field $\mathbb{R}$

Theorem 1. [29]

In the space $\mathbb{C}^{m\times n}$ over the field $\mathbb{R}$, an inner product can be defined as

$$\langle A, B \rangle = \mathrm{Re}[tr(A^H B)]. \qquad (2)$$

Proof.

(1) For $A, B \in \mathbb{C}^{m\times n}$, according to the properties of trace of a matrix one has

$$\langle A, B \rangle = \mathrm{Re}[tr(A^H B)] = \mathrm{Re}[tr(B^T \overline{A})] = \mathrm{Re}[\overline{tr(B^T \overline{A})}]$$
$$= \mathrm{Re}[tr(B^H A)] = \langle B, A \rangle$$

(2) For a real number a, and $A, B, C \in \mathbb{C}^{m\times n}$, one has

$$\langle aA, B \rangle = \mathrm{Re}[tr((aA)^H B)] = \mathrm{Re}[tr(aA^H B)] = \mathrm{Re}[a\, tr(A^H B)]$$
$$= a\ \mathrm{Re}[tr(A^H B)] = a \langle A, B \rangle$$

$$\langle A + B, C \rangle = \mathrm{Re}[tr((A+B)^H C)] = \mathrm{Re}[tr(A^H + B^H)C]$$
$$= \mathrm{Re}[tr(A^H C)] + \mathrm{Re}[tr(B^H C)] = \langle A, C \rangle + \langle B, C \rangle$$

(3) It is well-known that $tr(A^H A) \succ 0$ for all $x \neq 0$. Thus, $\langle A, A \rangle = \mathrm{Re}[tr(A^H A)] \succ 0$ for all $x \neq 0$.

According to definition 1, all the above argument reveals that the space $\mathbb{C}^{m\times n}$ over field $\mathbb{R}$ with the inner product defined by (2) is an inner product space.

Definition2. Frobenius Norm

The matrix norm of A induced by the inner product is Frobenius norm and denoted by $\|A\|$ $\|A\|^2 = \langle A, A \rangle = trace(A^H A)$.

3. Main Results

In this section, we propose an iterative solution to the complex matrix equation

$$\sum_{i=1}^{s} A_i V B_i + \sum_{j=1}^{t} C_j W D_j = \sum_{l=1}^{m} E_l \overline{V} F_l + G, \qquad (1)$$

where $A_i, C_j, E_l \in \mathbb{C}^{n\times n}$, $B_i, D_j, F_l \in \mathbb{C}^{p\times p}$ and $G \in \mathbb{C}^{n\times p}$ are given matrices, while $V, W \in \mathbb{C}^{n\times p}$ are matrices to be determined.

The solution the matrix equation (1) is based on the following algorithm.

Algorithm I (Finite Iterative Algorithm for (1))

1 Input $A_i, B_i, C_j, D_j E_l, F_l, G$

2 Chosen arbitrary matrices $V_1, W_1 \in \mathbb{C}^{n\times p}$;

3 Set

$$R_1 = G + \sum_{l=1}^{m} E_l \overline{V_1} F_l - \sum_{i=1}^{s} A_i V_1 B_i - \sum_{j=1}^{t} C_j W_1 D_j;$$

$$P_1 = \sum_{i=1}^{s} A_i^H R_1 B_i^H - \sum_{l=1}^{m} \overline{E_l}^H \overline{R_1} \overline{F_l}^H;$$

$$Q_1 = \sum_{j=1}^{t} C_j^H R_1 D_j^H;$$

$k := 1$

1 If $R_k = 0$, then stop; and V_k, W_k are the solution; else let $k := k + 1$ go to STEP 5.

2 Compute

$$V_{k+1} = V_k + \frac{\|R_k\|^2}{\|P_k\|^2 + \|Q_k\|^2} P_k;$$

$$W_{k+1} = W_k + \frac{\|R_k\|^2}{\|P_k\|^2 + \|Q_k\|^2} Q_k ;$$

$$R_{k+1} = G + \sum_{l=1}^{m} E_l \overline{V_{k+1}} F_l - \sum_{i=1}^{s} A_i V_{k+1} B_i - \sum_{j=1}^{t} C_j W_{k+1} D_j$$

$$= R_k - \frac{\|R_k\|^2}{\|P_k\|^2 + \|Q_k\|^2} [\sum_{i=1}^{s} A_i P_k B_i - \sum_{l=1}^{m} E_l \overline{P_k} F_l + \sum_{j=1}^{t} C_j Q_k D_j];$$

$$P_{k+1} = \sum_{i=1}^{s} A_i^H R_{k+1} B_i^H - \sum_{l=1}^{m} \overline{E_l}^H \overline{R_{k+1}} \overline{F_l}^H + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} P_k ;$$

$$Q_{k+1} = \sum_{j=1}^{t} C_j^H R_{k+1} D_j^H + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} Q_k ;$$

3 If $R_{k+1} = 0$;then stop ;else let $k := k+1$; go to STEP 5.

To prove the convergence property of Algorithm I, we first establish the following basic properties

Lemma 1

Suppose the matrix equation (1) is consistent and let V^*, W^* be its solution. Then, for any initial matrices V_1, W_1 , we have

$$tr[P_i^H(V^* - V_i) + Q_i^H(W^* - W_i)] + \overline{tr[P_i^H(V^* - V_i) + Q_i^H(W^* - W_i)]} = 2\|R_i\|^2 \quad (3)$$

Or, equivalently

$$\mathrm{Re}\{tr[P_i^H(V^* - V_i) + Q_i^H(W^* - W_i)]\} = \|R_i\|^2$$

Where the sequences $\{V_i\},\{P_i\},\{W_i\},\{Q_i\}$ and $\{R_i\}$ are generated by Algorithm I for $i = 1,2,\ldots$

Proof

We apply mathematical induction to prove the conclusion

For $i = 1$, from Algorithm I we have

$$tr[P_1^H(V^* - V_1) + Q_1^H(W^* - W_1)] = tr[(\sum_{i=1}^{s} A_i^H R_1 B_i^H - \sum_{l=1}^{m} \overline{E_l}^H \overline{R_1} \overline{F_l}^H)^H (V^* - V_1)$$

$$+ (\sum_{j=1}^{t} C_j^H R_1 D_j^H)^H (W^* - W_1)]$$

$$= tr[R_1^H \sum_{i=1}^{s} A_i V^* B_i + \overline{R_1}^H \sum_{l=1}^{m} \overline{E_l} V_1 \overline{F_l} + R_1^H \sum_{j=1}^{t} C_j W^* D_j$$

$$- R_1^H \sum_{i=1}^{s} A_i V_1 B_i - \overline{R_1}^H \sum_{l=1}^{m} \overline{E_l} V^* \overline{F_l} - R_1^H \sum_{j=1}^{t} C_j W_1 D_j]$$

In view that V^*, W^* are solutions of the matrix equation (1), it is easy that one can obtain from above relation

$$tr[P_1^H(V^* - V_1) + Q_1^H(W^* - W_1)] + \overline{tr[P_1^H(V^* - V_1) + Q_1^H(W^* - W_1)]}$$

$$= tr[R_1^H \sum_{i=1}^{s} A_i V^* B_i + \overline{R_1}^H \sum_{l=1}^{m} \overline{E_l} V_1 \overline{F_l} + R_1^H \sum_{j=1}^{t} C_j W^* D_j - R_1^H \sum_{i=1}^{s} A_i V_1 B_i - \overline{R_1}^H \sum_{l=1}^{m} \overline{E_l} V^* \overline{F_l}$$

$$- R_1^H \sum_{j=1}^{t} C_j W_1 D_j] + tr[\overline{R_1}^H \sum_{i=1}^{s} \overline{A_i} \overline{V^*} \overline{B_i} + R_1^H \sum_{l=1}^{m} E_l \overline{V_1} F_l + \overline{R_1}^H \sum_{j=1}^{t} \overline{C_j} \overline{W^*} \overline{D_j} - \overline{R_1}^H \sum_{i=1}^{s} \overline{A_i} \overline{V_1} \overline{B_i}$$

$$- R_1^H \sum_{l=1}^{m} E_l \overline{V^*} F_l - \overline{R_1}^H \sum_{j=1}^{t} \overline{C_j} \overline{W_1} \overline{D_j}]$$

$$= tr[R_1^H (\sum_{i=1}^{s} A_i V^* B_i + \sum_{j=1}^{t} C_j W^* D_j - \sum_{l=1}^{m} E_l \overline{V^*} F_l) - R_1^H (\sum_{i=1}^{s} A_i V_1 B_i + \sum_{j=1}^{t} C_j W_1 D_j - \sum_{l=1}^{m} E_l \overline{V_1} F_l)$$

$$+ \overline{R_1}^H (\sum_{i=1}^{s} \overline{A_i} \overline{V^*} \overline{B_i} + \sum_{j=1}^{t} \overline{C_j} \overline{W^*} \overline{D_j} - \sum_{l=1}^{m} \overline{E_l} V^* \overline{F_l}) - \overline{R_1}^H (\sum_{i=1}^{s} \overline{A_i} \overline{V_1} \overline{B_i} + \sum_{j=1}^{t} \overline{C_j} \overline{W_1} \overline{D_j} - \sum_{l=1}^{m} \overline{E_l} V_1 \overline{F_l})]$$

$$= tr[R_1^H (G + \sum_{l=1}^{m} E_l \overline{V_1} F_l - \sum_{i=1}^{s} A_i V_1 B_i - \sum_{j=1}^{t} C_j W_1 D_j) + \overline{R_1}^H (\overline{G} + \sum_{l=1}^{m} \overline{E_l} V_1 \overline{F_l} - \sum_{i=1}^{s} \overline{A_i} \overline{V_1} \overline{B_i} - \sum_{j=1}^{t} \overline{C_j} \overline{W_1} \overline{D_j}$$

$$= tr(R_1^H R_1) + tr(\overline{R_1}^H \overline{R_1}) = 2\|R_1\|^2$$

This implies that (3) holds for $i = 1$.

Assume that (3) holds for $i = k$. That is,

$$tr[P_k^H(V^* - V_k) + Q_k^H(W^* - W_k)] + \overline{tr[P_k^H(V^* - V_k) + Q_k^H(W^* - W_k)]} = 2\|R_k\|^2$$

Then we have to prove that the conclusion holds for $i = k+1$. It follows from Algorithm I that

$$tr[P_{k+1}^H(V^*-V_{k+1})+Q_{k+1}^H(W^*-W_{k+1})]=tr[(\sum_{i=1}^{s}A_i^H R_{k+1}B_i^H-\sum_{l=1}^{m}\overline{E_l}^H\,\overline{R_{k+1}}\,\overline{F_l}^H+\frac{\|R_{k+1}\|^2}{\|R_k\|^2}P_k)^H(V^*-V_{k+1})$$

$$+(\sum_{j=1}^{t}C_j^H R_{k+1}D_j^H+\frac{\|R_{k+1}\|^2}{\|R_k\|^2}Q_k)^H(W^*-W_{k+1})]$$

$$=tr[R_{k+1}^H\sum_{i=1}^{s}A_i(V^*-V_{k+1})B_i-\overline{R}_{k+1}^H\sum_{l=1}^{m}\overline{E_l}(V^*-V_{k+1})\,\overline{F_l}+R_{k+1}^H\sum_{j=1}^{t}C_j(W^*-W_{k+1})D_j]$$

$$+\frac{\|R_{k+1}\|^2}{\|R_k\|^2}[tr(P_k^H(V^*-V_k-\frac{\|R_k\|^2}{\|P_k\|^2+\|Q_k\|^2}P_k)+Q_k^H(W^*-W_k-\frac{\|R_k\|^2}{\|P_k\|^2+\|Q_k\|^2}Q_k)]$$

$$=tr[R_{k+1}^H\sum_{i=1}^{s}A_i(V^*-V_{k+1})B_i-\overline{R}_{k+1}^H\sum_{l=1}^{m}\overline{E_l}(V^*-V_{k+1})\,\overline{F_l}+R_{k+1}^H\sum_{j=1}^{t}C_j(W^*-W_{k+1})D_j]$$

$$+\frac{\|R_{k+1}\|^2}{\|R_k\|^2}[tr(P_k^H(V^*-V_k)+Q_k^H(W^*-W_k))]-\frac{\|R_{k+1}\|^2}{\|R_k\|^2}[\frac{\|R_k\|^2}{\|P_k\|^2+\|Q_k\|^2}(\|P_k\|^2+\|Q_k\|^2)]$$

$$=tr[R_{k+1}^H\sum_{i=1}^{s}A_i(V^*-V_{k+1})B_i-\overline{R}_{k+1}^H\sum_{l=1}^{m}\overline{E_l}(V^*-V_{k+1})\,\overline{F_l}+R_{k+1}^H\sum_{j=1}^{t}C_j(W^*-W_{k+1})D_j]$$

$$+\frac{\|R_{k+1}\|^2}{\|R_k\|^2}[tr(P_k^H(V^*-V_k)+Q_k^H(W^*-W_k))]-\|R_{k+1}\|^2 \tag{4}$$

In view that V^*,W^* is a solution of matrix equation (1), with this relation and (4) one has

$$tr[P_{k+1}^H(V^*-V_{k+1})+Q_{k+1}^H(W^*-W_{k+1})]+\overline{tr[P_{k+1}^H(V^*-V_{k+1})+Q_{k+1}^H(W^*-W_{k+1})]}=$$

$$=tr[R_{k+1}^H\sum_{i=1}^{s}A_i(V^*-V_{k+1})B_i-\overline{R}_{k+1}^H\sum_{l=1}^{m}\overline{E_l}(V^*-V_{k+1})\,\overline{F_l}+R_{k+1}^H\sum_{j=1}^{t}C_j(W^*-W_{k+1})D_j]$$

$$+\frac{\|R_{k+1}\|^2}{\|R_k\|^2}[tr(P_k^H(V^*-V_k)+Q_k^H(W^*-W_k))]-2\|R_{k+1}\|^2$$

$$\overline{+tr[R_{k+1}^H\sum_{i=1}^{s}A_i(V^*-V_{k+1})B_i-\overline{R}_{k+1}^H\sum_{l=1}^{m}\overline{E_l}(V^*-V_{k+1})\,\overline{F_l}+R_{k+1}^H\sum_{j=1}^{t}C_j(W^*-W_{k+1})D_j]}$$

$$+\frac{\|R_{k+1}\|^2}{\|R_k\|^2}\overline{[tr(P_k^H(V^*-V_k)+Q_k^H(W^*-W_k))]}$$

$$=tr[R_{k+1}^H(\sum_{i=1}^{s}A_iV^*B_i+\sum_{j=1}^{t}C_jW^*D_j-\sum_{l=1}^{m}\overline{E_l}V^*\,\overline{F_l}-\sum_{i=1}^{s}A_iV_{k+1}B_i-\sum_{j=1}^{t}C_jW_{k+1}D_j+\sum_{l=1}^{m}\overline{E_l}V_{k+1}\,\overline{F_l})]$$

$$+\frac{\|R_{k+1}\|^2}{\|R_k\|^2}[tr(P_k^H(V^*-V_k)+Q_k^H(W^*-W_k))-2\|R_{k+1}\|^2$$

$$\overline{+tr[R_{k+1}^H(\sum_{i=1}^{s}A_iV^*B_i+\sum_{j=1}^{t}C_jW^*D_j-\sum_{l=1}^{m}\overline{E_l}V^*\,\overline{F_l}-\sum_{i=1}^{s}A_iV_{k+1}B_i-\sum_{j=1}^{t}C_jW_{k+1}D_j+\sum_{l=1}^{m}\overline{E_l}V_{k+1}\,\overline{F_l})]}$$

$$\overline{+tr(P_k^H(V^*-V_k)+Q_k^H(W^*-W_k))}]$$

$$= tr[R_{k+1}^H(G+\sum_{l=1}^{m} E_l \overline{V_{k+1}} F_l - \sum_{i=1}^{s} A_i V_{k+1} B_i - \sum_{j=1}^{t} C_j W_{k+1} D_j)] - 2\|R_{k+1}\|^2$$

$$+ \overline{tr[R_{k+1}^H(G+\sum_{l=1}^{m} E_l \overline{V_{k+1}} F_l - \sum_{i=1}^{s} A_i V_{k+1} B_i - \sum_{j=1}^{t} C_j W_{k+1} D_j)} + \frac{\|R_{k+1}\|^2}{\|R_k\|^2}(2\|R_k\|^2)$$

$$= tr(R_{k+1}^H R_{k+1}) + \overline{tr(R_{k+1}^H R_{k+1})} + 2\|R_{k+1}\|^2 - 2\|R_{k+1}\|^2$$

$$= 2\|R_{k+1}\|^2$$

Hence relation (3) holds by principle of induction.

Lemma 2

Suppose that the matrix equation (1) is consistent and the sequences $\{R_i\},\{P_i\}$ and $\{Q_i\}$ are generated by Algorithm I with any initial matrices V_1, W_1, such that $R_i \neq 0$ for all $i = 1,2,\ldots,k$ then,

$$\mathrm{Re}\{ trace(R_j^H R_i) \} = 0 \tag{5}$$

and $\mathrm{Re}\{trace(P_j^H P_i + Q_j^H Q_i)\} = 0$ for

$$i, j = 1,2,\ldots,k, \quad i \neq j. \tag{6}$$

Proof

We apply mathematical induction

Step 1: We prove that

$$tr(R_{i+1}^H R_i) = 0 \tag{7}$$

and $tr(P_{i+1}^H P_i + Q_{i+1}^H Q_i) = 0$ for $i = 1,2,\ldots,k$. (8)

First from Algorithm I we have

$$R_{k+1} = G + \sum_{l=1}^{m} E_l \overline{V_{k+1}} F_l - \sum_{i=1}^{s} A_i V_{k+1} B_i - \sum_{j=1}^{t} C_j W_{k+1} D_j$$

$$= G + \sum_{l=1}^{m} E_l \overline{(V_k + \frac{\|R_k\|^2}{\|P_k\|^2 + \|Q_k\|^2} P_k)} F_l - \sum_{i=1}^{s} A_i (V_k + \frac{\|R_k\|^2}{\|P_k\|^2 + \|Q_k\|^2} P_k) B_i$$

$$- \sum_{j=1}^{t} C_j (W_k + \frac{\|R_k\|^2}{\|P_k\|^2 + \|Q_k\|^2} Q_k) D_j$$

$$= G + \sum_{l=1}^{m} E_l \overline{V_k} F_l - \sum_{i=1}^{s} A_i V_k B_i - \sum_{j=1}^{t} C_j W_k D_j + \frac{\|R_k\|^2}{\|P_k\|^2 + \|Q_k\|^2}(\sum_{l=1}^{m} E_l \overline{P_k} F_l - \sum_{i=1}^{s} A_i P_k B_i - \sum_{j=1}^{t} C_j Q_k D_j)$$

$$= R_k - \frac{\|R_k\|^2}{\|P_k\|^2 + \|Q_k\|^2}[\sum_{i=1}^{s} A_i P_k B_i - \sum_{l=1}^{m} E_l \overline{P_k} F_l + \sum_{j=1}^{t} C_j Q_k D_j]; \tag{9}$$

For $i = 1$, it follows from (9) that

$$tr(R_2^H R_1) = tr((R_1 - \frac{\|R_1\|^2}{\|P_1\|^2 + \|Q_1\|^2}[\sum_{i=1}^{s} A_i P_1 B_i - \sum_{l=1}^{m} E_l \overline{P_1} F_l + \sum_{j=1}^{t} C_j Q_1 D_j])^H R_1)$$

$$= tr(R_1^H R_1 - \frac{\|R_1\|^2}{\|P_1\|^2 + \|Q_1\|^2}[P_1^H \sum_{i=1}^{s} A_i^H R_1 B_i^H - \overline{P_1}^H \sum_{l=1}^{m} E_l^H R_1 F_l^H + Q_1^H \sum_{j=1}^{t} C_j^H R_1 D_j^H]).$$

From this last relation one has

$$tr(R_2^H R_1)+\overline{tr(R_2^H R_1)}=2\|R_1\|^2-\frac{\|R_1\|^2}{\|P_1\|^2+\|Q_1\|^2}tr[P_1^H\sum_{i=1}^{s}A_i^H R_1 B_i^H-\overline{P_1}^H\sum_{l=1}^{m}E_l^H R_1 F_l^H+Q_1^H\sum_{j=1}^{t}C_j^H R_1 D_j^H]$$

$$-\frac{\|R_1\|^2}{\|P_1\|^2+\|Q_1\|^2}tr[\overline{P_1^H}\sum_{i=1}^{s}\overline{A_i^H}\,\overline{R_1}\,\overline{B_i^H}+\overline{Q_1^H}\sum_{j=1}^{t}\overline{C_j^H}\,\overline{R_1}\,\overline{D_j^H}-P_1^H\sum_{l=1}^{m}\overline{E_l^H}\,\overline{R_1}\,\overline{F_l^H}]$$

$$=2\|R_1\|^2-\frac{\|R_1\|^2}{\|P_1\|^2+\|Q_1\|^2}tr[P_1^H(\sum_{i=1}^{s}A_i^H R_1 B_i^H-\sum_{l=1}^{m}\overline{E_l}^H\,\overline{R_1}\,\overline{F_l}^H)+Q_1^H\sum_{j=1}^{t}C_j^H R_1 D_j^H$$

$$+\overline{P_1^H}(\sum_{i=1}^{s}\overline{A_i^H}\,\overline{R_1}\,\overline{B_i^H}-\sum_{l=1}^{m}E_l^H R_1 F_l^H)+\overline{Q_1^H}\sum_{j=1}^{t}\overline{C_j^H}\,\overline{R_1}\,\overline{D_j^H}]$$

$$=2\|R_1\|^2-\frac{\|R_1\|^2}{\|P_1\|^2+\|Q_1\|^2}tr[P_1^H P_1+Q_1^H Q_1+\overline{P_1^H}\,\overline{P_1}+\overline{Q_1^H}\,\overline{Q_1}]$$

$$=2\|R_1\|^2-\frac{\|R_1\|^2}{\|P_1\|^2+\|Q_1\|^2}[2\|P_1\|^2+2\|Q_1\|^2]=0$$ This implies that (7) is satisfied for $i=1$.

From Algorithm I we have

$$tr(P_2^H P_1+Q_2^H Q_1)=tr[(\sum_{i=1}^{s}A_i^H R_2 B_i^H-\sum_{l=1}^{m}\overline{E_l}^H\,\overline{R_2}\,\overline{F_l}^H+\frac{\|R_2\|^2}{\|R_1\|^2}P_1)^H P_1+(\sum_{j=1}^{t}C_j^H R_2 D_j^H+\frac{\|R_2\|^2}{\|R_1\|^2}Q_1)^H Q_1]$$

$$=tr[R_2^H\sum_{i=1}^{s}A_i P_1 B_i-\overline{R_2^H}\sum_{l=1}^{m}\overline{E_l}P_1\overline{F_l}+\frac{\|R_2\|^2}{\|R_1\|^2}P_1^H P_1+R_2^H\sum_{j=1}^{t}C_j Q_1 D_j+\frac{\|R_2\|^2}{\|R_1\|^2}Q_1^H Q_1]$$

From this last relation one has

$$tr(P_2^H P_1+Q_2^H Q_1)+\overline{tr(P_2^H P_1+Q_2^H Q_1)}=tr[R_2^H\sum_{i=1}^{s}A_i P_1 B_i-\overline{R_2^H}\sum_{l=1}^{m}\overline{E_l}P_1\overline{F_l}+R_2^H\sum_{j=1}^{t}C_j Q_1 D_j]$$

$$+\frac{\|R_2\|^2}{\|R_1\|^2}tr(P_1^H P_1+\overline{P_1^H}\,\overline{P_1}+Q_1^H Q_1+\overline{Q_1^H}\,\overline{Q_1})$$

$$+\overline{tr[R_2^H\sum_{i=1}^{s}A_i P_1 B_i-\overline{R_2^H}\sum_{l=1}^{m}\overline{E_l}P_1\overline{F_l}+R_2^H\sum_{j=1}^{t}C_j Q_1 D_j]}$$

$$=tr[R_2^H(\sum_{i=1}^{s}A_i P_1 B_i+\sum_{j=1}^{t}C_j Q_1 D_j-\sum_{l=1}^{m}E_l\overline{P_1}F_l)]+2\frac{\|R_2\|^2}{\|R_1\|^2}(\|P_1\|^2+\|Q_1\|^2)$$

$$+\overline{tr[R_2^H(\sum_{i=1}^{s}A_i P_1 B_i+\sum_{j=1}^{t}C_j Q_1 D_j-\sum_{l=1}^{m}E_l\overline{P_1}F_l)]}$$

$$=\frac{\|P_1\|^2+\|Q_1\|^2}{\|R_1\|^2}[tr(R_2^H(R_1-R_2))+\overline{tr(R_2^H(R_1-R_2))}]+2\frac{\|R_2\|^2}{\|R_1\|^2}(\|P_1\|^2+\|Q_1\|^2)$$

$$=-\frac{\|P_1\|^2+\|Q_1\|^2}{\|R_1\|^2}[2\|R_2\|^2]+2\frac{\|R_2\|^2}{\|R_1\|^2}(\|P_1\|^2+\|Q_1\|^2)=0$$

This implies that (8) is satisfied for $i=1$.

Assume (7) and (8) hold for $i=k-1$. From (9) and applying mathematical induction assumption, from Algorithm I we have

$$tr(R_{k+1}^H R_k)+\overline{tr(R_{k+1}^H R_k)}=tr[(R_k-\frac{\|R_k\|^2}{\|P_k\|^2+\|Q_k\|^2}[\sum_{i=1}^s A_iP_kB_i-\sum_{l=1}^m E_l\overline{P_k}F_l+\sum_{j=1}^t C_jQ_kD_j])^H R_k]$$
$$+\overline{tr[(R_k-\frac{\|R_k\|^2}{\|P_k\|^2+\|Q_k\|^2}[\sum_{i=1}^s A_iP_kB_i-\sum_{l=1}^m E_l\overline{P_k}F_l+\sum_{j=1}^t C_jQ_kD_j])^H R_k]}$$
$$=2\|R_k\|^2-\frac{\|R_k\|^2}{\|P_k\|^2+\|Q_k\|^2}[tr[P_k^H(P_k-\frac{\|R_k\|^2}{\|R_{k-1}\|^2}P_{k-1})+Q_k^H(Q_k-\frac{\|R_k\|^2}{\|R_{k-1}\|^2}Q_{k-1})]$$
$$+\overline{tr[P_k^H(P_k-\frac{\|R_k\|^2}{\|R_{k-1}\|^2}P_{k-1})+Q_k^H(Q_k-\frac{\|R_k\|^2}{\|R_{k-1}\|^2}Q_{k-1})]}]$$
$$=2\|R_k\|^2-\frac{\|R_k\|^2}{\|P_k\|^2+\|Q_k\|^2}[2(\|P_k\|^2+\|Q_k\|^2)-\frac{\|R_k\|^2}{\|R_{k-1}\|^2}[tr(P_k^H P_{k-1}+Q_k^H Q_{k-1})$$
$$+\overline{tr(P_k^H P_{k-1}+Q_k^H Q_{k-1})}]=0$$

Thus, (7) holds for $i=k$.

Also, from Algorithm I we have

$$tr(P_{k+1}^H P_k+Q_{k+1}^H Q_k)+\overline{tr(P_{k+1}^H P_k+Q_{k+1}^H Q_k)}=tr[(\sum_{i=1}^s A_i^H R_{k+1}B_i^H-\sum_{l=1}^m \overline{E_l}^H\overline{R_{k+1}}\overline{F_l}^H+\frac{\|R_{k+1}\|^2}{\|R_k\|^2}P_k)^H P_k$$
$$+(\sum_{j=1}^t C_j^H R_{k+1}D_j^H+\frac{\|R_{k+1}\|^2}{\|R_k\|^2}Q_k)^H Q_k]+\overline{tr[(\sum_{j=1}^t C_j^H R_{k+1}D_j^H+\frac{\|R_{k+1}\|^2}{\|R_k\|^2}Q_k)^H Q_k}$$
$$\overline{+(\sum_{i=1}^s A_i^H R_{k+1}B_i^H-\sum_{l=1}^m \overline{E_l}^H\overline{R_{k+1}}\overline{F_l}^H+\frac{\|R_{k+1}\|^2}{\|R_k\|^2}P_k)^H P_k}]$$

$$=tr[R_{k+1}^H(\sum_{i=1}^s A_iP_kB_i-\sum_{l=1}^m \overline{E_l}\overline{P_k}\overline{F_l}+\sum_{j=1}^t C_jQ_kD_j)+2\frac{\|R_{k+1}\|^2}{\|R_k\|^2}(\|P_k\|^2+\|Q_k\|^2)$$
$$\overline{+R_{k+1}^H(\sum_{i=1}^s A_iP_kB_i-\sum_{l=1}^m \overline{E_l}\overline{P_k}\overline{F_l}+\sum_{j=1}^t C_jQ_kD_j)}]$$

$$= tr[R_{k+1}^H(\frac{\|P_k\|^2+\|Q_k\|^2}{\|R_k\|^2}(R_k-R_{k+1})]+2\frac{\|R_{k+1}\|^2}{\|R_k\|^2}(\|P_k\|^2+\|Q_k\|^2)$$

$$+\overline{tr[R_{k+1}^H(\frac{\|P_k\|^2+\|Q_k\|^2}{\|R_k\|^2}(R_k-R_{k+1})]}$$

$$=\frac{\|P_k\|^2+\|Q_k\|^2}{\|R_k\|^2}[tr(R_{k+1}^H R_k)+\overline{tr(R_{k+1}^H R_k)}-2\|R_{k+1}\|^2]+2\frac{\|R_{k+1}\|^2}{\|R_k\|^2}(\|P_k\|^2+\|Q_k\|^2)$$

$$=\frac{\|P_k\|^2+\|Q_k\|^2}{\|R_k\|^2}[-2\|R_{k+1}\|^2]+2\frac{\|R_{k+1}\|^2}{\|R_k\|^2}(\|P_k\|^2+\|Q_k\|^2)=0$$

This implies that (7) and (8) hold for $i=k$.

Hence, relation (7) and (8) hold for all $1\le i\le k$

Step2: we want to show that

$$\mathrm{Re}(tr(R_{i+l}^H R_i))=0 \tag{10}$$

$$\text{and } \mathrm{Re}(tr(P_{i+l}^H P_i+Q_{i+l}^H Q_i))=0 \tag{11}$$

hold for integer $l\ge 1$. We will prove this conclusion by induction. The case of $l=1$ has been proven in Step 1. Now we assume that (7) and (8) holds for $l\le s, s\ge 1$. The aim is to show

$$\mathrm{Re}(tr(R_{i+s+1}^H R_i))=0 \tag{12}$$

$$\text{and } \mathrm{Re}(tr(P_{i+s+1}^H P_i+Q_{i+s+1}^H Q_i))=0 \tag{13}$$

First we prove the following

$$\mathrm{Re}(tr(R_{s+1}^H R_0))=0 \tag{14}$$

$$\text{and } \mathrm{Re}(tr(P_{s+1}^H P_0+Q_{s+1}^H Q_0))=0 \tag{15}$$

By using Algorithm I, from (9) and induction assumption we have

$$tr(R_{s+1}^H R_0)+\overline{tr(R_{s+1}^H R_0)}=tr[(R_s-\frac{\|R_s\|^2}{\|P_s\|^2+\|Q_s\|^2}[\sum_{i=1}^{s}A_iP_sB_i-\sum_{l=1}^{m}E_l\overline{P_s}F_l+\sum_{j=1}^{t}C_jQ_sD_j])^H R_0]$$

$$+\overline{tr[(R_s-\frac{\|R_s\|^2}{\|P_s\|^2+\|Q_s\|^2}[\sum_{i=1}^{s}A_iP_sB_i-\sum_{l=1}^{m}E_l\overline{P_s}F_l+\sum_{j=1}^{t}C_jQ_sD_j])^H R_0]}$$

$$=tr(R_s^H R_0)+\overline{tr(R_s^H R_0)}-\frac{\|R_s\|^2}{\|P_s\|^2+\|Q_s\|^2}tr([\sum_{i=1}^{s}A_iP_sB_i-\sum_{l=1}^{m}E_l\overline{P_s}F_l+\sum_{j=1}^{t}C_jQ_sD_j]^H R_0)$$

$$\overline{-\frac{\|R_s\|^2}{\|P_s\|^2+\|Q_s\|^2}tr([\sum_{i=1}^{s}A_iP_sB_i-\sum_{l=1}^{m}E_l\overline{P_s}F_l+\sum_{j=1}^{t}C_jQ_sD_j]^H R_0)}$$

$$=-\frac{\|R_s\|^2}{\|P_s\|^2+\|Q_s\|^2}[tr(P_s^H[\sum_{i=1}^{s}A_i^H R_0B_i^H-\sum_{l=1}^{m}\overline{E_l^H}\,\overline{R_0}\,\overline{F_l^H}])+tr(Q_s^H\sum_{j=1}^{t}C_j^H R_0D_j^H)$$

$$+\overline{tr(P_s^H[\sum_{i=1}^{s}A_i^H R_0B_i^H-\sum_{l=1}^{m}\overline{E_l^H}\,\overline{R_0}\,\overline{F_l^H}])}+\overline{tr(Q_s^H\sum_{j=1}^{t}C_j^H R_0D_j^H)}]$$

$$=-\frac{\|R_s\|^2}{\|P_s\|^2+\|Q_s\|^2}[tr(P_s^H P_0+Q_s^H Q_0)+\overline{tr(P_s^H P_0+Q_s^H Q_0)}]=0$$

And

$$tr(P_{s+1}^H P_0 + Q_{s+1}^H Q_0) + \overline{tr(P_{s+1}^H P_0 + Q_{s+1}^H Q_0)}$$

$$= tr[(\sum_{i=1}^{s} A_i^H R_{s+1} B_i^H - \sum_{l=1}^{m} \overline{E_l}^H \overline{R_{s+1}} \overline{F_l}^H + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} P_s)^H P_0 + (\sum_{j=1}^{t} C_j^H R_{s+1} D_j^H + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} Q_s)^H Q_0]$$

$$+ \overline{tr[(\sum_{j=1}^{t} C_j^H R_{s+1} D_j^H + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} Q_s)^H Q_0]} + \overline{tr[(\sum_{i=1}^{s} A_i^H R_{s+1} B_i^H - \sum_{l=1}^{m} \overline{E_l}^H \overline{R_{s+1}} \overline{F_l}^H + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} P_s)^H P_0]}$$

$$= tr[R_{s+1}^H (\sum_{i=1}^{s} A_i P_0 B_i - \sum_{l=1}^{m} E_l \overline{P_0} F_l + \sum_{j=1}^{t} C_j Q_0 D_j) + \overline{R_{s+1}^H (\sum_{i=1}^{s} A_i P_0 B_i - \sum_{l=1}^{m} E_l \overline{P_0} F_l + \sum_{j=1}^{t} C_j Q_0 D_j)}]$$

$$+ \frac{\|R_{s+1}\|^2}{\|R_s\|^2} [tr(P_s^H P_0 + Q_s^H Q_0) + \overline{tr(P_s^H P_0 + Q_s^H Q_0)}]$$

$$= tr[R_{s+1}^H (\sum_{i=1}^{s} A_i P_0 B_i - \sum_{l=1}^{m} E_l \overline{P_0} F_l + \sum_{j=1}^{t} C_j Q_0 D_j) + \overline{R_{s+1}^H (\sum_{i=1}^{s} A_i P_0 B_i - \sum_{l=1}^{m} E_l \overline{P_0} F_l + \sum_{j=1}^{t} C_j Q_0 D_j)}]$$

$$= \frac{\|P_1\|^2 + \|Q_1\|^2}{\|R_1\|^2} [tr(R_{s+1}^H (R_0 - R_1)) + \overline{tr(R_{s+1}^H (R_0 - R_1))}] = 0$$

Then both (14) and (15) are holds

From Algorithm I and (9), induction assumption one has

$$tr(P_{i+s+1}^H P_i + Q_{i+s+1}^H Q_i) + \overline{tr(P_{i+s+1}^H P_i + Q_{i+s+1}^H Q_i)} = tr[(\sum_{i=1}^{s} A_i^H R_{i+s+1} B_i^H - \sum_{l=1}^{m} \overline{E_l}^H \overline{R_{i+s+1}} \overline{F_l}^H + \frac{\|R_{i+s+1}\|^2}{\|R_{i+s}\|^2} P_{i+s})^H P_i$$

$$+ (\sum_{j=1}^{t} C_j^H R_{i+s+1} D_j^H + \frac{\|R_{i+s+1}\|^2}{\|R_{i+s}\|^2} Q_{i+s})^H Q_i] + tr\overline{[(\sum_{j=1}^{t} C_j^H R_{i+s+1} D_j^H + \frac{\|R_{i+s+1}\|^2}{\|R_{i+s}\|^2} Q_{i+s})^H Q_i]}$$

$$\overline{+ (\sum_{i=1}^{s} A_i^H R_{i+s+1} B_i^H - \sum_{l=1}^{m} \overline{E_l}^H \overline{R_{i+s+1}} \overline{F_l}^H + \frac{\|R_{i+s+1}\|^2}{\|R_{i+s}\|^2} P_{i+s})^H P_i]}$$

$$= tr[(\sum_{i=1}^{s} A_i^H R_{i+s+1} B_i^H - \sum_{l=1}^{m} \overline{E_l}^H \overline{R_{i+s+1}} \overline{F_l}^H)^H P_i + (\sum_{j=1}^{t} C_j^H R_{i+s+1} D_j^H)^H Q_i] + \overline{tr[(\sum_{j=1}^{t} C_j^H R_{i+s+1} D_j^H)^H Q_i]}$$

$$+ \overline{tr[(\sum_{i=1}^{s} A_i^H R_{i+s+1} B_i^H - \sum_{l=1}^{m} \overline{E_l}^H \overline{R_{i+s+1}} \overline{F_l}^H)^H P_i]} + \frac{\|R_{i+s+1}\|^2}{\|R_{i+s}\|^2} [tr(P_{i+s}^H P_i + Q_{i+s}^H Q_i) + \overline{tr(P_{i+s}^H P_i + Q_{i+s}^H Q_i)}]$$

$$= tr[R_{i+s+1}^H (\sum_{i=1}^{s} A_i P_i B_i - \sum_{l=1}^{m} E_l \overline{P_i} F_l + \sum_{j=1}^{t} C_j Q_i D_j) + \overline{R_{i+s+1}^H (\sum_{i=1}^{s} A_i P_i B_i - \sum_{l=1}^{m} E_l \overline{P_i} F_l + \sum_{j=1}^{t} C_j Q_i D_j)}]$$

$$= \frac{\|P_i\|^2 + \|Q_i\|^2}{\|R_i\|^2} [tr(R_{i+s+1}^H (R_i - R_{i+1}) + \overline{R_{i+s+1}^H (R_i - R_{i+1})})]$$

$$= \frac{\|P_i\|^2 + \|Q_i\|^2}{\|R_i\|^2} [tr(R_{i+s+1}^H R_i + \overline{R_{i+s+1}^H R_i})] \tag{16}$$

In addition, from (9) it can be shown that

$$
\begin{aligned}
tr(R_{i+s+1}^H R_i) + \overline{tr(R_{i+s+1}^H R_i)} &= tr[(R_{i+s} - \frac{\|R_{i+s}\|^2}{\|P_{i+s}\|^2 + \|Q_{i+s}\|^2}[\sum_{i=1}^{s} A_i P_{i+s} B_i - \sum_{l=1}^{m} E_l \overline{P_{i+s}} F_l + \sum_{j=1}^{t} C_j Q_{i+s} D_j])^H R_i] \\
&\quad + \overline{tr[(R_{i+s} - \frac{\|R_{i+s}\|^2}{\|P_{i+s}\|^2 + \|Q_{i+s}\|^2}[\sum_{i=1}^{s} A_i P_{i+s} B_i - \sum_{l=1}^{m} E_l \overline{P_{i+s}} F_l + \sum_{j=1}^{t} C_j Q_{i+s} D_j])^H R_i]} \\
&= tr(R_{i+s}^H R_i) + \overline{tr(R_{i+s}^H R_i)} - \frac{\|R_{i+s}\|^2}{\|P_{i+s}\|^2 + \|Q_{i+s}\|^2} tr([\sum_{i=1}^{s} A_i P_{i+s} B_i - \sum_{l=1}^{m} E_l \overline{P_{i+s}} F_l + \sum_{j=1}^{t} C_j Q_{i+s} D_j]^H R_i) \\
&\quad - \overline{\frac{\|R_{i+s}\|^2}{\|P_{i+s}\|^2 + \|Q_{i+s}\|^2} tr([\sum_{i=1}^{s} A_i P_{i+s} B_i - \sum_{l=1}^{m} E_l \overline{P_{i+s}} F_l + \sum_{j=1}^{t} C_j Q_{i+s} D_j]^H R_i)} \\
&= -\frac{\|R_{i+s}\|^2}{\|P_{i+s}\|^2 + \|Q_{i+s}\|^2}[tr(P_{i+s}^H [\sum_{i=1}^{s} A_i^H R_i B_i^H - \sum_{l=1}^{m} \overline{E_l^H}\,\overline{R_i}\,\overline{F_l^H}]) + tr(Q_{i+s}^H \sum_{j=1}^{t} C_j^H R_i D_j^H) \\
&\quad + \overline{tr(P_{i+s}^H [\sum_{i=1}^{s} A_i^H R_i B_i^H - \sum_{l=1}^{m} \overline{E_l^H}\,\overline{R_i}\,\overline{F_l^H}])} + \overline{tr(Q_{i+s}^H \sum_{j=1}^{t} C_j^H R_i D_j^H)}] \\
&= -\frac{\|R_{i+s}\|^2}{\|P_{i+s}\|^2 + \|Q_{i+s}\|^2}[tr(P_{i+s}^H [P_i - \frac{\|R_i\|^2}{\|R_{i-1}\|^2} P_{i-1}]) + \overline{tr(P_{i+s}^H [P_i - \frac{\|R_i\|^2}{\|R_{i-1}\|^2} P_{i-1}])} \\
&\quad + tr(Q_{i+s}^H [Q_i - \frac{\|R_i\|^2}{\|R_{i-1}\|^2} Q_{i-1}]) + \overline{tr(Q_{i+s}^H [Q_i - \frac{\|R_i\|^2}{\|R_{i-1}\|^2} Q_{i-1}])}] \\
&= -\frac{\|R_{i+s}\|^2}{\|P_{i+s}\|^2 + \|Q_{i+s}\|^2} \frac{\|R_i\|^2}{\|R_{i-1}\|^2}[tr(P_{i+s}^H P_{i-1} + Q_{i+s}^H Q_{i-1}) + \overline{tr(P_{i+s}^H P_{i-1} + Q_{i+s}^H Q_{i-1})}]. \qquad (17)
\end{aligned}
$$

Repeating (16) and (17), one can easily obtain for certain α and β

$$tr(P_{i+s+1}^H P_i + Q_{i+s+1}^H Q_i) + \overline{tr(P_{i+s+1}^H P_i + Q_{i+s+1}^H Q_i)} = \alpha[tr(P_{s+1}^H P_0 + Q_{s+1}^H Q_0) + \overline{tr(P_{s+1}^H P_0 + Q_{s+1}^H Q_0)}]$$

and

$$tr(R_{i+s+1}^H R_i) + \overline{tr(R_{i+s+1}^H R_i)} = \beta[tr(R_{s+1}^H R_0) + \overline{tr(R_{s+1}^H R_0)}].$$

Combining these two relations with (12) and (13) implies that (8) and (9) hold for $l = s+1$. From step (1) and (2) the conclusion holds by the principle of induction.

Remark

Lemma 1 implies that if there exist a positive number i such that $P_i = 0$ and $Q_i = 0$ but $R_j \neq 0$, then the matrix equation (1) is inconsistent.

With the above two lemmas, we have the following theorem.

Theorem 2[19]

If the matrix equation (1) is consistent, then a solution can be obtained within finite iteration steps by using Algorithm I for any initial matrices V_1, W_1.

4 Numerical Examples

In this section, we present a numerical example to illustrate the application of our proposed methods.

Example

In this example we illustrate our theoretical result of algorithm I for solving the matrix Equation

$$\sum_{i=1}^{s} A_i V B_i + \sum_{j=1}^{t} C_j W D_j = \sum_{l=1}^{m} E_l \overline{V} F_l + G$$

As a special case

$$A_1 V B_1 + A_2 V B_2 + C_1 W D_1 + C_2 W D_2 = E_1 \overline{V} F_1 + G$$

Given

$$A_1 = \begin{bmatrix} 3+i & -i & -3i \\ 1+2i & 4+i & 0 \\ 2i & 1+i & 4 \end{bmatrix}, A_2 = \begin{bmatrix} 2+3i & 1+2i & 0 \\ -1+3i & 3-i & 2i \\ 4 & -3i & 0 \end{bmatrix}$$

$$,E_1=\begin{bmatrix}-1-i & 2 & 5\\ -1 & 2+i & 3+i\\ 4+i & 2i & -3i\end{bmatrix}$$

$$C_1=\begin{bmatrix}1+2i & -i & 1\\ 4 & 2i & i\\ 3-i & 3+i & 1+i\end{bmatrix},$$

$$C_2=\begin{bmatrix}i & 1-i & 1+i\\ -3i & 3 & 3i\\ 2-i & 1+i & 2\end{bmatrix},B_1=\begin{bmatrix}-1+i & 2+i\\ 3 & -2i\end{bmatrix}$$

$$B_2=\begin{bmatrix}2i & 4\\ 5i & 1+3i\end{bmatrix},D_1=\begin{bmatrix}1+3i & 2-i\\ 1+i & 1+2i\end{bmatrix},D_2=\begin{bmatrix}-2 & 1\\ 1+i & 2-2i\end{bmatrix}$$

$$,\ C=\begin{bmatrix}-56+20i & -9+53i\\ 71i & 51+3i\\ 13+14i & 74-14i\end{bmatrix},F_1=\begin{bmatrix}1+i & 0\\ 2i & -3i\end{bmatrix}$$

Taking $V_1=\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$ $,W_1=\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$. We apply Algorithm I to compute V_k,W_k.

After iterating 14 steps we obtain

$$V_{14}=\begin{bmatrix}0.9455-0.5720i & 0.8962+0.2947i\\ 1.4545-0.8987i & 1.7584-0.4302i\\ -0.4653+1.0047i & 0.2661-0.8237i\end{bmatrix}$$

$$W_{14}=\begin{bmatrix}1.6318+1.8856i & 0.0742-0.1201i\\ 0.9854-0.7156i & -0.1297-1.8948i\\ -0.2652-0.1812i & -1.0442-0.1023i\end{bmatrix}$$

which satisfy the matrix equation

$$A_1VB_1+A_2VB_2+C_1WD_1+C_2WD_2=E_1\overline{V}F_1+G$$

with the corresponding residual

$$\|R_{14}\|=\|G+E_1\overline{V}F_1-A_1VB_1-A_2VB_2-C_1WD_1-C_2WD_2\|=7.2584\times10^{-10}$$

The obtained results are presented in figure 1, where

$$r_k=\|R_k\| \qquad \text{(Residual)}$$

From Fig. 1, it is clear that the error δ_k is becoming smaller and approaches zero as iteration number k increases. This indicates that the proposed algorithm is effective and convergent.

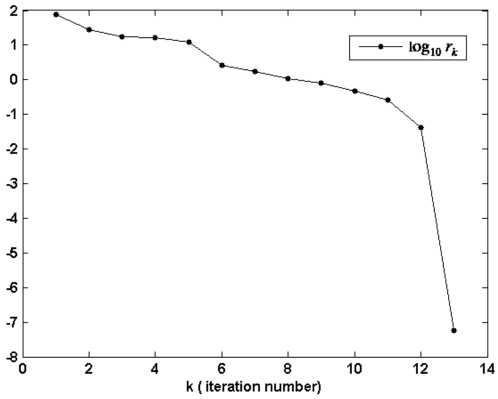

Fig. 1. *The residual and the relative error versus k (iteration number)*

5 Conclusions

Iterative solution for the general Sylvester-conjugate matrix equation $\sum_{i=1}^{s}A_iVB_i+\sum_{j=1}^{t}C_jWD_j=\sum_{l=1}^{m}E_l\overline{V}F_l+G$ is presented. We have proven that the iterative algorithm always converge to the solution for any initial matrices. We stated and proved some lemmas and theorems where the solutions are obtained. The obtained results show that the methods are very neat and efficient. The proposed methods are illustrated by numerical example. Example we tested using MATLAB to verify our theoretical results.

References

[1] Dehghan, M., Hajarian, M. (2008), An iterative algorithm for the reflexive solutions of the generalized coupled Sylvester matrix equations and its optimal approximation, Appl. Math. Comput., 202 , pp 571–588.

[2] Dehghan, M., Hajarian, M. (2008), An iterative algorithm for solving a pair of matrix equations $AYB=E,CYD=F$ over generalized Centro-symmetric matrices, Comput. Math. Appl. ,56, pp 3246–3260.

[3] Wang, Q.W. (2005), Bisymmetric and Centro symmetric solutions to systems of real quaternion matrix equations, Comput. Math. Appl., 49, pp 641–650.

[4] Wang, Q.W., Zhang, F. (2008), The reflexive re-nonnegative definite solution to a quaternion matrix equation, Electron. J. Linear Algebra ,17, pp88–101.

[5] Wang, Q.W., Zhang, H.S., Yu, S.W. (2008), On solutions to the quaternion matrix equation $AXB+CYD=E$, Electron. J. Linear Algebra, 17, 343–358.

[6] Ding,F. , Liu, P.X. , Ding, J. (2008) ,Iterative solutions of the generalized Sylvester matrix equations by using the hierarchical identification principle, Appl. Math. Comput. ,197 ,pp 41–50.

[7] Peng, X.Y. , Hu, X.Y, Zhang, L. (2007) , The reflexive and anti-reflexive solutions of the matrix equation $A^HXB=C$, J. Comput. Appl. Math. ,186, pp638–645.

[8] Ramadan, M. A., Abdel Naby, M A. , Bayoumi, A. M. E. (2009), On the explicit solution of the Sylvester and the yakubovich matrix equations, Math. Comput. Model. ,50,pp1400-1408.

[9] Wu, A. G., Duan, G.R., Yu, H.H. (2006), On solutions of the matrix equations $XF - AX = C$ and $XF - A\overline{X} = C$, Appl. Math. Comput. ,183 ,pp932–941.

[10] Zhou, B., Li, Z.Y., Duan, G.R., Wang, Y. (2009), Weighted least squares solutions to general coupled Sylvester matrix equations, J. Comput. Appl. Math. ,224,pp 759–776.

[11] Zhou, B., Duan, G.R. (2008), On the generalized Sylvester mapping and matrix equations, Systems Control Lett., 57 ,pp200–208.

[12] Ding,F., Chen, T. (2005) , Gradient based iterative algorithms for solving a class of matrix equations, IEEE Trans. Automat. Control ,50 ,pp1216–1221.

[13] Ding,F., Chen, T. (2005) , Hierarchical gradient-based identification of multivariable discrete-time systems, Automatica, 41, pp 315–325.

[14] Ding,F., Chen, T. (2005) , Iterative least squares solutions of coupled Sylvester matrix equations, Systems Control Lett. ,54 ,pp 95–107.

[15] Ding,F., Chen, T. (2006) , on iterative solutions of general coupled matrix equations, SIAM J. Control Optim. ,44, pp 2269–2284.

[16] Ding,F., Chen, T. (2005) , Hierarchical least squares identification methods for multivariable systems, IEEE Trans. Automat. Control ,50, pp 397–402.

[17] Wang, X., Wu, W. H. (2011), A finite iterative algorithm for solving the generalized (P, Q)-reflexive solution of the linear systems of matrix equations, Mathematical and Computer Modelling, 54,pp 2117-2131.

[18] Wu, A. G., Li, B. , Zhang, Y. ,Duan, G.R. (2011), Finite iterative solutions to coupled Sylvester-conjugate matrix equations. Applied Mathematical Modelling, , 35(3),pp 1065-1080.

[19] Wu, A. G., Lv, L., Hou, M.-Z. (2011), Finite iterative algorithms for extended Sylvester-conjugate matrix equation, Math. Comput. Model. ,54,pp2363-2384.

[20] Ramadan, M. A., Abdel Naby, M A. , Bayoumi, A. M. E. (2014), Iterative algorithm for solving a class of general Sylvester-conjugate Matrix equation $\sum_{i=1}^{s} A_i V + \sum_{j=1}^{t} B_j W = \sum_{l=1}^{m} E_l \overline{V} F_l + C$, J. Appl. Math. and Comput, 44(1-2),pp 99-11 8.