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# Julia and Mandelbrot Sets of the Gamma Function Using Lanczos Approximation

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**Abstract:** This work explores the Julia and Mandelbrot sets of the Gamma function by extending the function to the entire complex plane through analytic continuation and functional equations. Various Julia and Mandelbrot sets associated with the Gamma function are generated using the iterative function  $f_{\lambda}(z) = \Gamma(z) + \lambda$ , with different parameter  $\lambda$  – values. To produce an accurate result using the integral definition of the Gamma function, a large number of terms would have to be added during the numerical integration procedure; this makes computation of Gamma function a very difficult task. To overcome this challenge, the Lanczos approximation of the Gamma function which presents an efficient and easy way to compute algorithms for approximating the Gamma function to an arbitrary precision is used. The resulting images reveal that the fractal (chaotic) behaviour found in elementary functions is also found in the Gamma function. The chaotic nature of the Julia and Mandelbrot sets provides a way of understanding complexity in systems as well as just in shapes.

**Keywords:** Julia Set, Mandelbrot Set, Gamma Function, Lanczos Approximation, Complex Functions

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## 1. Introduction

Julia and its related Mandelbrot sets are typical examples of complex dynamical systems which exhibit fractal behaviour. Until the advent of modern computers, the complex dynamical systems enjoyed low patronage because of the enormous nature of their calculations which made them impractical for real use. Julia was famous in the 1920s, but his work on iteration was essentially forgotten until Benoit Mandelbrot brought it back to prominence in the 1970s through the fundamental computer experiment [1]; Mandelbrot was the first to use computer to produce graphic representation of complex dynamical system based on the polynomial and rational functions described by Gaston Julia [2]. Braverman [3] observed that Julia sets are some of the best known illustrations of a highly complicated chaotic system generated by a very simple mathematical process. The Mandelbrot sets

are concerned with the parameter plane only. They are fractals and contain some feature like those found in nature such as mountains, trees, lightning, waterfalls, etc.

The Julia and Mandelbrot set have been studied extensively over the years using elementary polynomial functions. In recent times, the study has been extended beyond the elementary functions to a special function (the Riemann zeta function) by [4]. This extension to a higher function was the first of its kind. [5] presented a construction of famous fractal images- Mandelbrot set and Julia set using 3D iterated function system which gives real look and feel of complex natural fractal images. Also [6] applied Jungck Ishikawa iteration to generate new relative superior Mandelbrot sets and relative superior Julia sets.

Our desire is to know how these sets behave with other special functions like the Gamma function. The Gamma function was only defined for real arguments. It was Carl

Friendrich Gauss who first studied Gamma function of complex variable. The integral form of the Gamma function is referred to as the second Eulerian integral. The Gamma function is an extension of the concept of factorial numbers. We can input (almost) any real or complex number into the Gamma function and find its value.

However, the definition integral is not very useful in computing the Gamma function; to produce an accurate result, an insanely high number of terms would have to be added during the numerical integration procedure. However, Stirling series, Spouge series and Lanczos approximation present efficient and easy way to compute algorithms for approximating the Gamma function to an arbitrary precision [7]. In order to produce the graphic representation of Julia sets and Mandelbrot sets, Lanczos Gamma approximation shall be used.

Based on the foregoing, we intend to explore Julia and Mandelbrot Sets under the Gamma function and represent them pictorially using MATLAB and also discuss the resulting representation.

## 2. Julia and Mandelbrot Sets

### 2.1. Julia Sets

Gaston Julia worked with Pierre Joseph Louis Fatou on multiple iterations of the complex quadratic map

$$f_\lambda(z) = z^2 + \lambda \quad z, \lambda \in \mathbb{C}. \tag{1}$$

Julia discovered a set of points whose  $n^{\text{th}}$  iterates,  $f_\lambda^n$  do not diverge to infinity as  $n \rightarrow \infty$ . This set of points form what is called filled-in Julia set whereas the boundary of this set is the Julia set. The *filled-in Julia set* of a polynomial  $f$ , denoted by  $B_\lambda$ , is the set of all points with bounded orbits under  $f$ . In symbols,

$$B_\lambda = \{z : |f_\lambda^n(z)| \frac{1}{2} \nrightarrow \infty \text{ as } n \rightarrow \infty\}.$$

The *Julia set* of  $f$ , denoted by  $J_\lambda$ , is the boundary of the filled-in Julia set. Julia sets are not restricted to the function (1). For any function Julia sets may be generated. These sets vary from function to function. Julia sets can be generally classified into connected and disconnected sets [8, 9].

### 2.2. Mandelbrot Sets

The Mandelbrot set  $M$  of a map  $f_\lambda(z)$ , for a chosen initial iteration value  $z = z_0$ , is the set of values of the complex parameter  $\lambda$  which in the iteration of  $f_\lambda(z)$  in (1) with the chosen initial value  $z_0$ , does not lie in the domain of attraction of complex infinity [4]. In other words,

$$M = \{\lambda : \lambda \in \mathbb{C}, \lim_{n \rightarrow \infty} f_\lambda^n(z) \frac{1}{2} \nrightarrow \infty\}.$$

Though the above definition is for polynomials and not the most general for non-polynomials, it is adopted here to study,

for a start, a particular one-parameter family of the iterated maps of the Gamma function, i.e., that of  $f_\lambda(z) = \Gamma(z) + \lambda$ .

From  $|f_\lambda^n(z)| \geq |z|(1+\epsilon)^n \rightarrow \infty$ ,

it can be determined whether  $\lambda$  is in Mandelbrot set [8]. A way of estimating whether  $f_\lambda(z)$  is bounded for a given  $\lambda$  is by checking if the first 50 to 100 iterations of the critical point  $z = 0$  stays within a circle of radius 2. By using a colouring scheme based on whether orbits escape this circle, a picture of  $M$  can be built.

## 3. Lanczos Gamma Approximation

The Gamma function  $\Gamma(z)$  is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad \Re(z) > 0 \tag{2}$$

with the fundamental recurrence relation

$$\Gamma(n+1) = n\Gamma(n) \tag{3}$$

The functional relation (3) can be used to find an analytic continuation of the Gamma function for  $\Re(z) \leq 0$  [10, 11].

However, the definition integral (2) is not very useful in computing the Gamma function. In order to produce the graphic representation of Julia sets and Mandelbrot sets, we will need the Lanczos Gamma approximation [12].

$$\Gamma(z+1) = \sqrt{2\pi} \left( z+r+\frac{1}{2} \right)^{z+\frac{1}{2}} e^{-(z+r+\frac{1}{2})} S_r(z) \tag{4}$$

where

$$S_r(z) = \frac{1}{2} a_0(r) + a_1(r) \frac{z}{z+1} + a_2(r) \frac{z(z-1)}{(z+1)(z+2)} + \dots \tag{5}$$

and

$$a_n(r) = \sum_{k=1}^n C(2n+1, 2k+1) \frac{\sqrt{2}}{\pi} \left( k - \frac{1}{2} \right)! x \left( k+r+\frac{1}{2} \right)^{-(k+\frac{1}{2})} e^{k+r+\frac{1}{2}} \tag{6}$$

with  $C(i, j)$  denoting the  $(i, j)^{\text{th}}$  element of the Chebyshev polynomial coefficient matrix which can be calculated recursively from the identity

$$C(1,1) = 1; \quad C(2,2) = 1$$

$$C(i,1) = -C(i-2,1) \quad i = 3, 4, \dots$$

$$C(i,j) = 2C(i-1,j-1) \quad i = j = 3, 4, \dots$$

$$C(i,j) = 2C(i-1,j-1) - C(i-2,j) \quad i > j = 2, 3, \dots$$

To derive (4), we use (7) and (8) below which are the Stirling's and Spouge's series respectively:

$$\Gamma(z+1) \approx \sqrt{2\pi z} \left( z^z e^{-z} \right) \quad \text{as } |z| \rightarrow \infty \tag{7}$$

$$\Gamma(z+1) = (z+a)^{z+\frac{1}{2}} e^{-(z+a)} (2\pi)^{1/2} \left[ c_0 + \sum_{k=1}^N \frac{c_k}{z+k} + \varepsilon(z) \right]. \quad (8)$$

From (2), let  $\alpha t \rightarrow t$ , then  $\alpha dt \rightarrow dt$

$$\therefore \Gamma(z+1) = \int_0^\infty (\alpha t)^z e^{-\alpha t} \alpha dt = \alpha^{z+1} \int_0^\infty t^z e^{-\alpha t} dt$$

Let  $\alpha = 1+\rho z$ ;  $\rho > 0$ , then

$$\begin{aligned} \Gamma(z+1) &= (1+\rho z)^{z+1} \int_0^\infty t^z e^{-(1+\rho z)t} dt \\ &= (1+\rho z)^{z+1} \int_0^\infty t^z e^{-\rho z t} e^{-t} dt \\ &= (1+\rho z)^{z+1} \int_0^\infty (te^{-\rho t})^z e^{-t} dt \end{aligned}$$

Introducing  $\left(\frac{e\rho}{e\rho}\right)^z$  into the equation and evaluating yields

$$\begin{aligned} \therefore \Gamma(z+1) &= \left(\frac{e\rho}{e\rho}\right)^z (1+\rho z)^{z+1} \int_0^\infty (te^{-\rho t})^z e^{-t} dt \\ &= (1+\rho z)^{z+1} (e\rho)^{-z} \int_0^\infty (\rho te^{-\rho t})^z e^{-t} dt \end{aligned}$$

Let  $v = e^{1-\rho t}$ , then  $dv = -\rho e^{1-\rho t} dt \Rightarrow dt = \frac{dv}{-\rho v}$ ,

and  $t = \frac{1}{\rho} - \frac{1}{\rho} \log v$ .

$$\begin{aligned} \Gamma(z+1) &= (1+\rho z)^{z+1} (e\rho)^{-z} \int_e^1 [v(1-\log v)]^z e^{-\left(\frac{1}{\rho} - \frac{1}{\rho} \log v\right)} \frac{dv}{-\rho v} \\ &= \left(\frac{1}{\rho} + z\right)^{z+1} \rho^{z+1} (e\rho)^{-z} \rho^{-1} \int_e^1 [v(1-\log v)]^z e^{-\left(\frac{1}{\rho} - \frac{1}{\rho} \log v\right)} \frac{dv}{v} \\ &= \left(\frac{1}{\rho} + z\right)^{z+1} e^{-\frac{1}{\rho}} \int_0^e [v(1-\log v)]^z v^{\frac{1}{\rho}-1} dv \end{aligned}$$

Let  $r = \frac{1}{\rho} - 1$ , then

$$\Gamma(z+1) = (z+r+1)^{z+1} e^{-(z+r+1)} \int_0^e [v(1-\log v)]^z v^r dv$$

Let  $v(1-\log v) = \cos^2\theta$ , then  $dv = \frac{2 \sin \theta \cos \theta}{\log v} d\theta$ ; when

$v = 0$ ,  $\theta = -\frac{\pi}{2}$  and when  $v = e$ ,  $\theta = \frac{\pi}{2}$ .

$$\begin{aligned} (z+1) &= (z+r+1)^{z+1} e^{-(z+r+1)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2\theta)^z v^r \frac{2 \sin \theta \cos \theta}{\log v} d\theta \\ &= \sqrt{2} (z+r+1)^{z+1} e^{-(z+r+1)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2z+\theta} \theta \frac{\sqrt{2} v^r \sin \theta}{\log v} d\theta \end{aligned}$$

Let  $z - \frac{1}{2} \rightarrow z$ , then

$$\Gamma\left(z + \frac{1}{2}\right) = P_r(z) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2z}\theta \left[ \frac{\sqrt{2} v^r \sin \theta}{\log v} \right] d\theta \quad (9)$$

where

$$P_r(z) = \sqrt{2} \left( z+r + \frac{1}{2} \right)^{z+\frac{1}{2}} e^{-\left(z+r+\frac{1}{2}\right)}$$

The integral in (9) is evaluated using Fourier series expansion to give the required result

$$\Gamma(z+1) = \sqrt{2\pi} \left( z+r + \frac{1}{2} \right)^{z+\frac{1}{2}} e^{-\left(z+r+\frac{1}{2}\right)} S_r(z).$$

### 4. Results

For the Gamma function  $\Gamma(z)$ , we can generate different Julia sets as the value of the parameter  $\lambda$  varies.

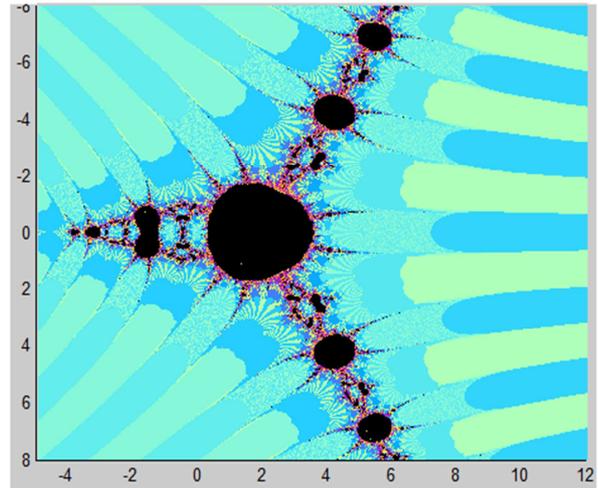


Figure 1. Julia Set of  $f_\lambda(z) = \Gamma(z) + \lambda$ ;  $\lambda = 0$ .

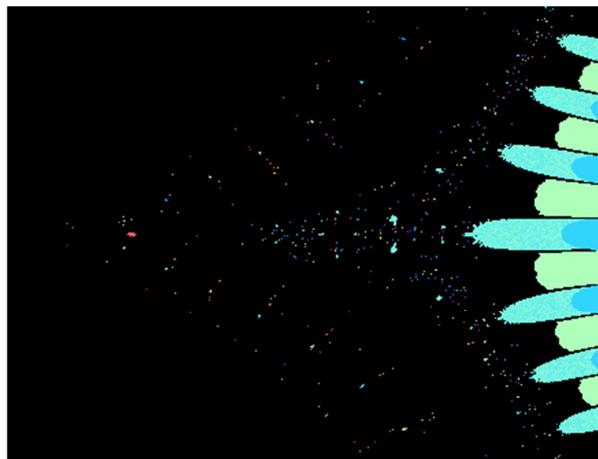


Figure 2. Julia Set for  $f_\lambda(z) = \Gamma(z) + \lambda$ ;  $\lambda = -0.621$ .

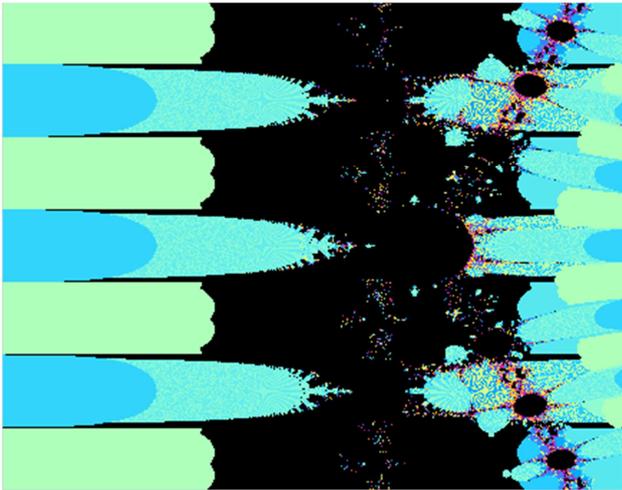


Figure 3. Julia Set of  $f_\lambda(z) = \Gamma(z) + e^{-z}$ .

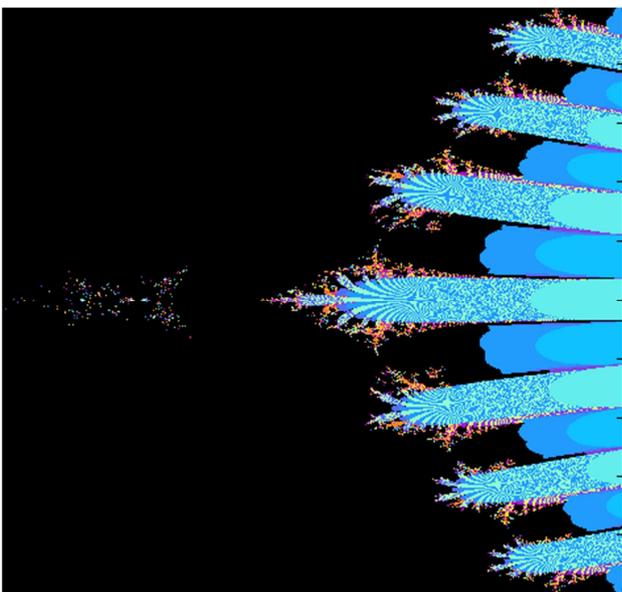


Figure 4. The Mandelbrot set for  $f_\lambda(z) = \Gamma(z) + \lambda$ .

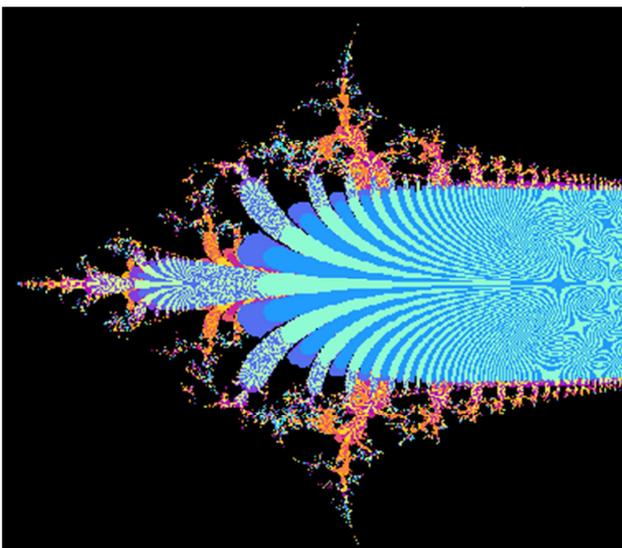


Figure 5. Magnification of the leading frond in Figure 4.

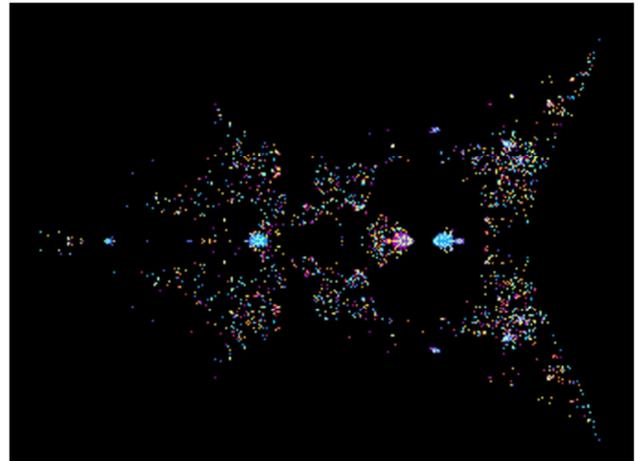


Figure 6. Zoom in of the galaxies on the left in Figure 4.

Fig. 1 shows the Julia set for  $f_\lambda(z) = \Gamma(z) + \lambda$  within  $-6.5 \leq \Re(z) \leq 8.5$  and  $-8.5 \leq \Im(z) \leq 8.5$ . In the figure is a horizontal mast with black patches together with spikes proportionate to their various sizes. The arms of the mast extend in opposite directions. The black portions represent the basins of attraction of the attractor of the fixed point at  $z = 1$  and those of attracting cycles. The Julia set is the boundaries around the mast i.e. the set of complex numbers  $z = x + iy$  whose orbits remain bounded after a number of iterations. The remaining part of the plane are coloured different shadings which represent the various rates at which the points escape to infinity, with the lighter shades having slower rates, and the darker shades, faster rates. There is a reflection symmetry about the  $\Re(z)$ -axis due to the complex conjugate property of an analytic complex function.

Two factors are primarily responsible for the change in the Julia set—the function and the parameter value. In Figure 2 showing the Julia set for  $f_\lambda(z) = \Gamma(z) + \lambda$  within  $-12.0 \leq \Re(z) \leq 8.5$  and  $-10.5 \leq \Im(z) \leq 10.5$ , the set is altered as the parameter value changes. Fig. 3, the Julia set for  $f_\lambda(z) = \Gamma(z) + e^{-z}$  within  $-12.0 \leq \Re(z) \leq 8.5$  and  $-10.5 \leq \Im(z) \leq 10.5$ , shows what happens when the function is altered. Instead of the function  $f_\lambda(z) = \Gamma(z) + \lambda$ , the function  $f_\lambda(z) = \Gamma(z) + e^{-z}$  is used.

For any function that a Julia set is generated, there is always a corresponding Mandelbrot set. It is noteworthy that, as infinitely many Julia sets are associated with a function, only one Mandelbrot set is connected to the function. In Fig. 4 showing the Mandelbrot set for  $f_\lambda(z) = \Gamma(z) + \lambda$  within  $-3.5 \leq \Re(z) \leq 8.0$  and  $-10.0 \leq \Im(z) \leq 10.0$ , the black region which forms the bulk of the plane is the basin of attraction of the fixed point and is regarded as the Mandelbrot set for  $f_\lambda(z) = \Gamma(z) + \lambda$ . In this figure, there are infinitely manyfronds of varying shadings. Each of them is studded with infinite number of smaller fronds all around its boundary which are its miniature copies. Fig. 5 and 6 are the magnifications of various parts of the Mandelbrot set for  $f_\lambda(z) = \Gamma(z) + \lambda$  in Fig. 4.

Unlike other functions where self-similarity can be spotted with the Mandelbrot set, the Gamma function seems to reveal the fractal of the complement of the Mandelbrot set. This does not imply that the Mandelbrot set has lost its fractal property in Gamma function because the boundary of the complement is also the boundary of the Mandelbrot set. This only implies that self-similarity is not only associated with the Mandelbrot set but also its complement.

In the elementary functions, Mandelbrot set is the set of parameters for which orbits of zero is bounded. With special functions, the Mandelbrot set is generated by the zeros of the function i.e. given the map  $f_\lambda(z) = \Gamma(z) + \lambda$  for which the initial iteration value  $z = z_0$  is a zero of the Gamma function  $\Gamma(z)$ , a value of the complex parameter  $\lambda$  belongs to the Mandelbrot set of  $f$  if the iterated value  $f_\lambda^n(z_0)$  does not tend to complex infinity as  $n \rightarrow \infty$  in the course of iteration.

## 5. Conclusion

The Gamma function was introduced and various Julia and Mandelbrot sets associated with it were generated. The Lanczos Gamma approximation formula was used to approximate values of the Gamma function resulting in the production of the images of these sets. This approximation was necessary because the built-in Gamma function in MATLAB R2010a does not take complex argument.

The images reveal that the chaotic behaviour found in the elementary functions are still present in the Gamma function. Although, these behaviour appears to be more conspicuous with the complement of the Julia and Mandelbrot sets. The chaotic nature of the Julia and Mandelbrot sets can provide a way of understanding complexity insystems" as well as just in shapes.

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