Applied and Computational Mathematics
2021; 10(5): 114-120
http://www.sciencepublishinggroup.com/j/acm
doi: 10.11648/j.acm.20211005.12
ISSN: 2328-5605 (Print); ISSN: 2328-5613 (Online)

A Knot Invariant Defined Based on the Skein Relation with Two Equations

Liu Weili*, **Lu Huimin**

Department of Basic Science, Dalian Naval Academy, Dalian, China

Email address:
648746489@qq.com (Liu Weili), 1456632645@qq.com (Lu Huimin)
*Corresponding author

To cite this article:
Liu Weili, Lu Huimin. A Knot Invariant Defined Based on the Skein Relation with Two Equations. *Applied and Computational Mathematics*. Vol. 10, No. 5, 2021, pp. 114-120. doi: 10.11648/j.acm.20211005.12

Received: September 9, 2021; **Accepted**: October 10, 2021; **Published**: October 16, 2021

Abstract: Knot theory is a branch of the geometric topology, the core question of knot theory is to explore the equivalence classification of knots; In other words, for a knot, how to determine whether the knot is an unknot; giving two knots, how to determine whether the two knots are equivalent. To prove that two knots are equivalent, it is necessary to turn one knot into another through the same mark transformation, but to show that two knots are unequal, the problem is not as simple as people think. We cannot say that they are unequal because we can't see the deformation between them. For the equivalence classification problem of knots, we mainly find equivalent invariants between knots. Currently, scholars have also defined multiple knot invariants, but they also have certain limitations, and even more difficult to understand. In this paper, based on existing theoretical results, we define a knot invariant through the skein relation with two equations. To prove this knot invariant, we define a function $f(L)$, and to prove $f(L)$ to be a homology invariant of a non-directed link, we need to show that it remains constant under the Reideminster moves. This article first defines the $f_k(L)$, the property of $f(L)$ is obtained by using the properties of $f_k(L)$. In the process of proof, the induction method has been used many times. The proof process is somewhat complicated, but it is easier to understand. And the common knot invariant is defined by one equation, which defining the knot invariant with two equations in this paper.

Keywords: Knot, Invariants, Reidemeister Moves, Skein Relation

1. Introduction

Topology is a branch of mathematics that is mainly studying the properties that objects remain unchanged after continuous transformations. And a more attractive area of topology is knot theory, there are two main reasons: One is because it studies real geometric phenomena in real life; the other is because it is mysterious to use different methods to study it thoroughly. Because of this, knot theory meets topology like differential geometry, number theory, algebraic geometry, matrix theory and group theory. Long long ago, people have already tie with rope, the most famous is: knot rope can remember, that is to record things in life by tying knots on the rope. knots can be seen everywhere in people's daily life, such as the popular Chinese knot during the Spring Festival in recent years, tying things with rope, sewing clothes, tying shoelaces, knitting sweaters and so on. On different occasions, people use different knots, so how to describe the knot mathematically? Although the famous allusion to the knot rope appeared before the invention, but from the mathematical perspective began in the 19th century, started by the German mathematician Calfried Gauss, he studied the nature of the electromagnetic field, found that the circle number between closed curves is related to the knot, this discovery laid a solid foundation for the study of knot theory, so the circle number became one of the main tools for scholars to study the knot. In 1867, Lord Kelvin regarded the atoms as the knot of the Etheric vortex, and one could classify the atoms with the aid of the classification of them. At the time, the hypothesis attracted many mathematicians, chemists, physicists to study knots, and knot theory emerged. A knot is the way a ring is embedded in 3 D space. Since any two knots are identical in the sense of homoderm, and we mainly study the way the curve is embedded in S^3, we do not consider the length, the hardness, the degree of bending and the thickness of the curve itself. The knot theory is mainly to explore the equivalence classification of knots [1]. Trefoil knots are the simplest extraordinary knots. To prove that two knots are equivalent, it is necessary to turn one knot into another through the same mark transformation. But to show that two knots are unequal, the problem is not as simple as people think. We cannot say that they are unequal because we can't see the deformation between them. For the equivalence classification problem of knots, we mainly find equivalent invariants between knots. In 1928, the Alexander of the United States discovered the first knot polynomial invariant in history, in 1969; British Conway modified Alexander polynomials to obtain the Conway polynomial [2]. In 1984, the New Zealand mathematician Jones obtained Jones polynomials when studying operator polynombras, Subsequently, many scholars have found some more general knot invariants, such as chain ring branch number in [3], A family of polynomial invariants for flat virtual knots in [4], Whitney towers and abelian invariants of knots in [5], An infinite-rank summand of knots with trivial Alexander polynomial in [6], A polynomial time knot polynomial in [7], Regional knot invariants in [8], A free-group valued invariant of free knots in [9], An invariant for colored bonded knots in [10], Multi-switches and virtual knot invariants in [11], Invariants of knot diagrams in [12], which provide a strong theoretical basis for distinguishing knots, but these invariants are still insufficient, which prompted people to continue to find new knot invariants, this paper construct a new one to provide a new method for the classification of knots.

2. Preliminary Data

Definition 2.1 An embedded K: $S^1 \to R^3$ in S^1 to R^3 is a knot.

The knot on the left in Figure 1 is the simplest knot, commonly known as an ordinary or unknot; the knot on the right is called a trefoil knot. Figure 2 are several examples of additional knots.

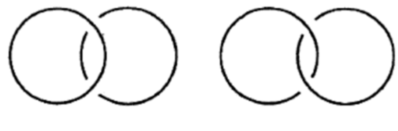
Figure 1. *The unknot and A trefoil knot.*

Figure 2. *Knots.*

A link is composed of finite knots in space, and they disintersect each other. If a link is composed of n nodes, it is called the link with n branches, and each knot forming the link is called a branch of the link. Obviously, the knot is a special link, and it has only one branch.

Obviously, in the plane, a link composed of finite multiple disjoint junctions must be a trivial link.

Below are several simple links:



Trivial 2-component link Hopf link

Whitehead link Borromean rings

Figure 3. *Links.*

Definition 2.2 [13] We say that two links L_1 and L_2 in S^3 are isotopic, written $L_1 \approx L_2$, if there exists an isotopy $F: S^3 \times I \to S^3$ such that $F_0 = Id$, $F_1(L_1) = L_2$.

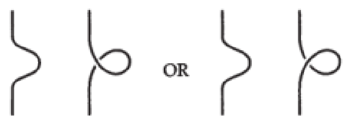
Definition 2.3 [14] If a projection diagram of the link K meets the following conditions:

(1) Only a finite points on the projection plot are overlapping points;
(2) This finite overlap points are the cross sections of the two arcs on the link;
(3) At each key, the image of the arc located at the knot above indicates a solid line, and the line disconnected at the secondary key indicates the image of the arc at the knot below;

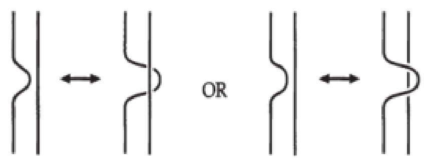
It is called a regular projection graph of the link K.

Theorem 2.1 [15] Two links diagrams are equivalent if and only if there are finite multiple Reidemeister moves, make one turn into the other.

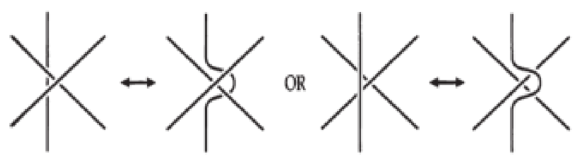
The three Reidemeister moves are performed as follows:



(R_1 is to eliminate or add a volume)



(R_2 is to add or eliminate a stacked binary)



(R_3 is a triangle change)

Figure 4. *Reidemeister moves.*

Definition 2.4 Let K be a knot, and if K has a projection graph on which a n intersection transformation yields an ordinary knot, and no junction transformation for any projection graph doing less than n times of K, the number of solution knots of K is called n. written u (K)=n.

Proposition 2.1 Any projection graph G, given a knot K so that its number of in tersections is that n, can always pass an intersection transformation of no more than n times and turn it into a projection graph of ordinary nodes.

3. Proposal of Polynomial Invariants

If L_+, L_-, L_0, L_∞ is the projection diagram representing the four links, they are exactly the same elsewhere except for the local difference of an intersection shown in Figure 5.

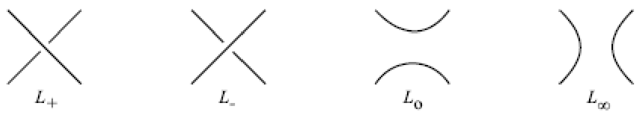


Figure 5. *An Local link projection diagram.*

Definition 3.1 [16] If an algebra is generated by a,b,c,a',b',c' $t_n, n=1,2,\cdots$, and the following three conditions are met:

(1) $a^2=1, a'^2=1, c=-ab, c'=-a'b'$.

(2) $(a'-a)b'=(a'-a)b=0, bb=bb', (aa'-1)bb=0$.

(3) $(-1+a+c)t_n+bt_{n+1}=0, n\geq 1$.

Then this algebra generates a Z-algebra, written Σ.

Theorem 2.1 Given an algebra Σ, there are homomorphic invariants with unique non-orientations of the links $f(L)\in\Sigma$, and there are the following two types of establishment:

(1) $f(T_n)=t_n$ (initial condition) (T_n is the trivial link with n branches).

(2) When the intersection comes from the same branch, $f(L_+)=af(L_-)+bf(L_0)+cf(L_\infty)$.

When the intersection comes from different branches, $f(L_+)=a'f(L_-)+b'f(L_0)+c'f(L_\infty)$.

4. Theorem Proving

Definition 4.1 Let L be a directional link diagram of n branches, $B=(b_1,\cdots,b_n)$ is the basis points of L, each branch takes a point and cannot be an intersection. If it goes back to b_1 from b_1, back to b_2 from b_2, and so on, back to b_n from b_n in the direction of L, If each intersection is the upper intersection, the L is a decreasing graph relative to the base B.

Definition 4.2 Relative to a basis point B, If advanced along the indicated direction, the first crossing of the intersection is through under it, hen say the intersection is a bad point relative to the base point B. The number of bad intersections is called the number of bad intersections, written $b(L)$.

Lemma 4.1 [17] L is a strand link diagram of K intersections and has a fixed branching order $L_1, L_2, \cdots, L_n$. then either L has a trivial branch or you can choose a set of basis points $B=(b_1,\cdots,b_i,\cdots,b_n)$ to drop relative to the basis points B and the number of intersections becomes a less than K intersection after a series of Reidemeister transformations.

To prove the theorem 2.1, A function $f(L)$ is defined. To prove that $f(L)$ is a homomorphic invariant of a non-orientations links,, Only Certificate $f(L)$ remains unchanged under the Reideminster transformation. for this purpose, first define $f_k(L)$, The property of $f(L)$ is obtained by using the properties of $f_k(L)$. The specific proof is given below (for writing convenience, the same branch when the intersection comes from the same branch, and the different branches when the intersection comes from different branches):

Prove First, give any orientation of the link.

We demonstrate this using a mathematical induction method on $cr(L)$ ($cr(L)$ represents crossing points in the diagram).

I. For a diagram L of n components with $cr(L)=0$, we put $f_0(L)=t_n$

II. when $k\geq 0, cr(L)\leq k$, The assumption $f_k(L)\in A$ is defined and $f_k(L)$ satisfied:

$$f_k(U_n)=t_n$$

(for Un being a descending diagram of n components).

$$f_k(L_+)=af_k(L_-)+bf_k(L_0)+cf_k(L_\infty)$$

(Same branchs)

$$f_k(L_+)=a'f_k(L_-)+b'f_k(L_0)+c'f_k(L_\infty)$$

(Different branches)

$$f_k(L)=f_k(R(L))$$

(R is Reideminste rmoves that makes $cr(R(L))\leq k$)

Since $f_k(L)$ is uniquely determined by the above several properties, and each knot projection graph can decompose into a linear combination of a series of descent graphs, so a function $f_k(L)$ is defined, written $f(L)=f_k(L)$.

Due to the $f_k(L)$ being met:

$$f_k(L_+)=af_k(L_-)+bf_k(L_0)+cf_k(L_\infty)(Same\ branchs)$$
$$f_k(L_+)=a'f_k(L_-)+b'f_k(L_0)+c'f_k(L_\infty)(Different\ branches)$$

So on the cross point $cr(L)\leq k$, $f(L)$ satisfied:

$$f(L_+)=af(L_-)+bf(L_0)+cf(L_\infty)(Same\ branchs)$$
$$f(L_+)=a'f(L_-)+b'f(L_0)+c'f(L_\infty)(Different\ branches)$$

If R is a Reidemeister move on a diagram L, then

$$cr(R(L))\leq k=cr(L)+2$$

So

$$f\left(R\left(L\right)\right)=f_k\left(R\left(L\right)\right)$$

$$f\left(L\right)=f_k\left(L\right)$$

by properties of $f_k(L)$: $f_k\left(L\right)=f_k\left(R\left(L\right)\right)$.

So

$$f(L)=f(\mathrm{R}(L))$$

When $cr(L)\le k$, $f(L)$ stay unchanged under Reidemeiste rmoves.

So U_n is isotopy to T_n.

For $f_k(U_n)=t_n$,

So

$$f_k(T_n)=t_n.$$

Therefore

$$f(T_n)=t_n$$

III. Determine a function $f_B(L)$ related to the basis point for a given knot projection graph and the corresponding basis point.

When $cr(L)\le k$, we put $f_B(L)=f_k(L)$

If U_n is a descending diagram with respect to B, we put $f_B(U_n)=t_n$ (n denotes the number of components).

Next, the $b(L)$ is summarized below:

Assume that $f_B(L)$ is defifined for $cr(L)\le k+1$ and $b(L)<t$, when $b(L)=t$

If p is the first bad crossing of L,

$$\begin{cases} f_B(L_+)=af_B(L_-)+bf_B(L_0)+cf_B(L_\infty)(\textit{Same branchs}) \\ f_B(L_+)=a'f_B(L_-)+b'f_B(L_0)+c'f_B(L_\infty)(\textit{Different branches}) \end{cases}$$

The other points of the following certificate also meet the relationship formula:

$$\begin{cases} f_B(L_+)=af_B(L_-)+bf_B(L_0)+cf_B(L_\infty)(\textit{Same branchs}) \\ f_B(L_+)=a'f_B(L_-)+b'f_B(L_0)+c'f_B(L_\infty)(\textit{Different branches}) \end{cases}$$

To facilitate writing, L_+^p indicates cross points p is L_+; L_{++}^{pq} indicates cross points p is L_+, q is L_+; L_{+-}^{pq} indicates cross points p is L_+, q is L_- and so on.

We will represent the considered intersections with the p. We think about the case $b(L^P{}_+)>b(L_-^P)$

If p is an intersection from the same branch (Other cases are similarly discussed).

The $b(L_-^P)$ is summarized below:

when $b(L_-^P)=0$, there is $b(L_+^P)$=1 p is the only bad intersection of L_+^P.

such that $b(L_-^P)<t, t\ge 1$.

$$b(L_-^P)\text{=t}\ge 1,\ b(L_+^P)\ge 2$$

If q is the first bad intersection on L_+^P.

I Assume that $q=p$, the conclusion was established

II Assume $q\ne p$, if q is L_+ and the intersection is on the same branch.

$$f_B(L_+^P)=f_B(L_{++}^{pq})=af_B(L_{+-}^{pq})+bf_B(L_{+0}^{pq})+cf_B(L_{+\infty}^{pq})$$

And $b(L_{--}^{pq})<t$ $cr(L_{+0}^{pq})\le k, cr(L_{+\infty}^{pq})\le k$

Judging from the above induction and the main induction method:

$$f_B(L_{+-}^{pq})=af_B(L_{--}^{pq})+bf_B(L_{0-}^{pq})+cf_B(L_{\infty-}^{pq})$$
$$f_B(L_{+0}^{pq})=af_B(L_{-0}^{pq})+bf_B(L_{00}^{pq})+cf_B(L_{\infty 0}^{pq})$$
$$f_B(L_{+\infty}^{pq})=a'f_B(L_{-\infty}^{pq})+b'f_B(L_{0\infty}^{pq})+c'f_B(L_{\infty\infty}^{pq})$$

And by algebras Σ, the conditions (1)(2) are easy to launch $a'c=ac$ $b'c=bc$ $c'c=cc$

And $b(L_{--}^{pq})<t$, $cr(L_0^p)\le k, cr(L_\infty^p)\le k$

So

$$f_B(L_-^p)=f_B(L_{-+}^{pq})=af_B(L_{--}^{pq})+bf_B(L_{-0}^{pq})+cf_B(L_{-\infty}^{pq})$$
$$f_B(L_0^p)=f_B(L_{0+}^{pq})=af_B(L_{0-}^{pq})+bf_B(L_{00}^{pq})+cf_B(L_{0\infty}^{pq})$$
$$f_B(L_\infty^p)=f_B(L_{\infty+}^{pq})=a'f_B(L_{\infty-}^{pq})+b'f_B(L_{\infty 0}^{pq})+c'f_B(L_{\infty\infty}^{pq})$$

Therefore, it is available using the above formula:

$$\begin{aligned} f_B(L_+^p) &= af_B(L_{+-}^{pq})+bf_B(L_{+0}^{pq})+cf_B(L_{+\infty}^{pq}) \\ &= af_B(L_{-+}^{pq})+bf_B(L_{0+}^{pq})+cf_B(L_{\infty+}^{pq}) \\ &= af_B(L_-^p)+bf_B(L_0^p)+cf_B(L_\infty^p) \end{aligned}$$

So

$$f_B(L_+^p)=af_B(L_-^p)+bf_B(L_0^p)+cf_B(L_\infty^p)$$

q is L_+ and Crossing are similarly probable when on different branches.

Therefore, $f_B(L)$ satisfied Relationships:

$$\begin{cases} f_B(L_+)=af_B(L_-)+bf_B(L_0)+cf_B(L_\infty)(\textit{Same branchs}) \\ f_B(L_+)=a'f_B(L_-)+b'f_B(L_0)+c'f_B(L_\infty)(\textit{Different branches}) \end{cases}$$

We will show that when the branch order does not change, $f_B(L)$ does not depend on the selection of the base points. Take b_i' after b_i and have only one intersection in the middle, say p, between b_i and b_i'.

Make $B'=(b_1,\cdots,b_i',\cdots,b_n)$

Suppose p is the type L_+, and the type L_- can be proved similarly.

The induction method is used on $\beta(L)=\max(b(L),b'(L))$

1. Assume $\beta(L)=0$, then $b(L)=0=b'(L)$, then L and L' are descending with respect to both choices of base points, So

$$f_B(L)=t_n=f_{B'}(L).$$

2. Assume that $\beta(L)=1$

(1) When $b(L)=1, b'(L)=0$, we can know that L is descending relative to b'.

$$f_{B'}(L)=t_n,$$

$$f_B(L)=f_B(L_+^p)=af_B(L_-^p)+bf_B(L_0^p)+cf_B(L_\infty^p)$$

(a) If L_∞^p branch more than one, $f_B(L)=(a+b)t_n+ct_{n+1}$ so

$$f_B(L)=t_n=f_{B'}(L)$$

(b) If L_0^p branch more than one, then

$$f_B(L)=(a+c)t_n+bt_{n+1}=t_n=f_{B'}(L)$$

(2) $b(L)=0$, $b'(L)=1$, a similar proof may be made.

(3) $b'(L)=b(L)=1$, p is the intersection on the different branches.

$$f_B(L)=a'f_B(L_-)+b'f_B(L_0)+c'f_B(L_\infty)$$
$$f_{B'}(L)=a'f_{B'}(L_-)+b'f_{B'}(L_0)+c'f_{B'}(L_\infty)$$

and $\beta(L_-^q)<\beta(L)$, $cr(L_0^q)\le k$, $cr(L_\infty^q)\le k$

So

$$f_B(L_0^q)=f_{B'}(L_0^q),$$

$$f_B(L_-^q)=f_{B'}(L_-^q),$$

$$f_B(L_\infty^q)=f_{B'}(L_\infty^q)$$

Therefore

$$f_B(L)=f_{B'}(L)$$

3. $\beta(L)=t>1$, if q is L_+ and the intersection q is derived from the same branch (Other cases may be similar).

$$f_B(L_+^q)=af_B(L_-^q)+bf_B(L_0^q)+cf_B(L_\infty^q)$$
$$f_{B'}(L_+^q)=af_{B'}(L_-^q)+bf_{B'}(L_0^q)+cf_{B'}(L_\infty^q)$$

and $\beta(L_-^q)<\beta(L)$,

So

$$f_B(L_-^q)=f_{B'}(L_-^q),$$

$$cr(L_0^q)\le k,\ cr(L_\infty^q)\le k$$

Therefore

$$f_B(L_0^q)=f_{B'}(L_0^q),$$

$$f_B(L_\infty^q)=f_{B'}(L_\infty^q)$$

$$f_B(L)=f_{B'}(L)$$

Therefore, we written $\tilde{f}(L)=f_B(L)\cdot$

The following certificate remains unchanged under the Reidemeister moves.

Let L be a link diagram that keeps the branching order unchanged, R is the Reidemeister moves on L

If $cr(L)\le k+1, cr(R(L))\le k+1$,

We use induction on $b(L)$:

(1) $b(L)=0$, so $b(R(L))=0$, Since the number of branches is unchanged, so

$$\tilde{f}(L)=\tilde{f}(R(L))$$

(2) $b(L)<t$, $\tilde{f}(L)=\tilde{f}(R(L))$

(3) $b(L)=t$,

(a) If a bad intersection point p is not involved in the Reidemeister move, then:

Suppose p is the type L_+ and the intersection is from the same branch, other cases are similar. It can be seen from the hypothesis:

$$\tilde{f}(L_-^p)=\tilde{f}(R(L_-^p))$$

$$\tilde{f}(L_0^p)=\tilde{f}(R(L_0^p))$$

$$\tilde{f}(L_\infty^p)=\tilde{f}(R(L_\infty^p))$$

and

$$\tilde{f}(L)=\tilde{f}(L_+^p)=a\tilde{f}(L_-^p)+b\tilde{f}(L_0^p)+c\tilde{f}(L_\infty^p)$$
$$\tilde{f}(R(L_+^p))=a\tilde{f}(R(L_-^p))+b\tilde{f}(R(L_0^p))+c\tilde{f}(R(L_\infty^p))$$
$$\tilde{f}(R(L))=\tilde{f}(R(L_+^p))$$

So

$$\tilde{f}(L)=\tilde{f}(R(L))$$

(b) There is no other bad point except for the bad point involved in the Reidemeister moves.

For the third Reidemeister move, if p is the intersection point from the same branch and p is L_+.

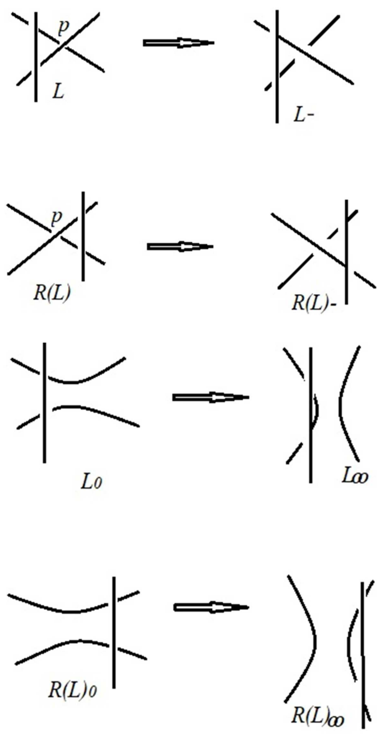


Figure 6. *The corresponding Reidemeister move.*

$$\tilde{f}(L)=\tilde{f}(L_+^p)=a\tilde{f}(L_-^p)+b\tilde{f}(L_0^p)+c\tilde{f}(L_\infty^p)$$
$$\tilde{f}(R(L)_+^p)=a\tilde{f}(R(L)_-^p)+b\tilde{f}(R(L)_0^p)+c\tilde{f}(R(L)_\infty^p)$$
$$\tilde{f}(R(L))=\tilde{f}(R(L)_+^p)$$

After a single R_3 transformation, $R(L_-)=R(L)_-$

After a single R_2 transformation, $R(R(L_\infty))=R(L)_\infty$

So

$$\tilde{f}(L^p)=\tilde{f}(R(L)^p)$$

For the first Reidemeister move, the basis point can always be transformed to make it good.

For the second Reidemeister move, there is only one case that cannot make the bad point better. That is, the intersections are from different branches and the intersection types are different. As shown in Figure 7.

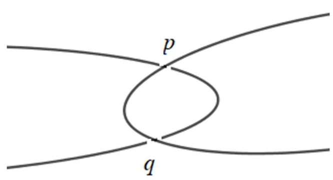


Figure 7. *Exceptional case.*

For this case, it is easily established by transformation and known conditions.

To sum up, $\tilde{f}(L)$ remains constant under the Reidemeister moves. So it can be ordered $f_{k+1}(L)=\tilde{f}(L)$.

Now we prove the independence of $\tilde{f}(L)$ of the branch order.

For a strand link diagram $L(cr(L)\le k+1)$ and fixed base points $B=(b_1,\cdots,b_i,b_{i+1},\cdots,b_n)$ and $B=(b_1,\cdots,b_{i+1},b_i,\cdots,b_n)$, we can derive $f_B(L)=f_{B'}(L)$.

We use induction on $b(L)$:

(1) $b(L)$ =0, it is established by lemma 4.1.

(2) If established at $b(L)<t$

(3) $b(L)=t$, Let p be a bad intersection.

If p is L_+, $b(L_-)<t$

By induction:

$$f_B(L_-)=f_{B'}(L_-),$$

$$cr(L_0)\le k,$$

$$cr(L_\infty)\le k$$

Therefore

$$f_{\mathrm{B}}(L)=f_{B'}(L).$$

Since the disassembly relation is independent of orientation, and the polynomial invariant of each chain loop is finally represented as some linear combination of decreasing knot graphs, it is easy to know that this polynomial invariant is independent of orientation, that is, the invariant is the same trace invariant of the non-orientation link.

This completes the proof.

5. Conclusion

In this paper, knot invariants by defining a system of split equations containing two equations, while common knot invariants are defined by a split equation which is a new method. This paper demonstrates the newly defined polynomial invariants by adopting multiple induction; Although the proof process is complicated, it is easy to understand, Compared to its proof, its practical application is somewhat difficult, which requires us to study later.

Fund Project

Scientific Research Development Fund of Dalian Naval Academy.

References

[1] ROLFSEN D. Knots and links [M]. Berkeley: Publish or Perish, 1976.

[2] ADAMS C C. The Knot Book [J]. American Mathematical Society Providence Ri, 1994: xiv, 307.

[3] Jiang Leping, Huang Fan Yanli. Figure 8.8.4 Number of chain loop branches of the lattice [J]. Journal of Hunan Institute of Technology: Natural Science Edition, 2012 (4): 17~18.

[4] Kyeonghui Lee, Young Ho Im, Sera Kim. A family of polynomial invariants for flat virtual knots [J]. Topology and its Applications, 2019, 264.

[5] Jae Choon Cha, Kent E. Orr, Mark Powell. Whitney towers and abelian invariants of knots [J]. Mathematische Zeitschrift, 2020, 294 (1).

[6] Min Hoon Kim, Kyungbae Park. An infinite-rank summand of knots with trivial Alexander polynomial [J]. Journal of Symplectic Geometry, 2019, 16 (6).

[7] Bar Natan Dror, van der Veen Roland. A polynomial time knot polynomial [J]. Proceedings of the American Mathematical Society, 2019, 147 (1).

[8] Zhiqing Yang. Regional knot invariants [J]. Journal of Knot Theory and Its Ramifications, 2017, 26 (6).

[9] Manturov Vassily Olegovich. A free-group valued invariant of free knots [J]. Journal of Knot Theory and Its Ramifications, 2021, 30 (06).

[10] Gabrovšek Boštjan. An invariant for colored bonded knots [J]. Studies in Applied Mathematics, 2021, 146 (3).

[11] Bardakov Valeriy, Nasybullov Timur. Multi-switches and virtual knot invariants [J]. Topology and its Applications, 2020.

[12] Joel Hass; Tahl Nowik. Invariants of knot diagrams [J]. Mathematische Annalen. 2008 (1).

[13] Boyd Rachael, Hepworth Richard, Patzt Peter The homology of the Brauer algebras [J] Selecta Mathematica, 2021, 27 (5).

[14] Agnese Barbensi, Daniele Celoria. The Reidemeister graph is a complete knot invariant [J]. Algebraic & Geometric Topology, 2020, 20 (2).

[15] Michael Polyak. Minimal generating sets of Reidemeister moves [J]. Quantum Topology. 2010 (4).

[16] Vyacheslav Futorny, João Schwarz Algebras of invariant differential operators [J] Journal of Algebra, 2020, 542.

[17] PRZYTYCKI J H. Survey on recent invariants on classical knot theory [J]. Eprint Arxiv, 2008.