Applied and Computational Mathematics
2021; 10(6): 138-145
http://www.sciencepublishinggroup.com/j/acm
doi: 10.11648/j.acm.20211006.12
ISSN: 2328-5605 (Print); ISSN: 2328-5613 (Online)

Asymptotic Behavior of Multivariate Extremes Geometric Type Random Variables

Frédéric Béré[1,*] , Kpèbbèwèrè Cédric Somé[2], Remi Guillaume Bagré[3], Pierre Clovis Nitiéma[4]

[1]Department of Mathematics, Institute of Sciences, Ouagadougou, Burkina Faso
[2]Department of Mathematics, Virtual University of Burkina Faso, Ouagadougou, Burkina Faso
[3]Department of Mathematics, Norbert ZONGO University, Koudougou, Burkina Faso
[4]Department of Mathematics of Decision, Thomas SANKARA University, Ouagadougou, Burkina Faso

Email address:

berefrederic@gmail.com (F. Béré), cedrickpebsom@yahoo.fr (K. C. Somé), baremgui@yahoo.fr (R. G. Bagré), pnitiema@gmail.com (P. C. Nitiéma)
*Corresponding author

To cite this article:

Frédéric Béré, Kpèbbèwèrè Cédric Somé, Remi Guillaume Bagré, Pierre Clovis Nitiéma. Asymptotic Behavior of Multivariate Extremes Geometric Type Random Variables. *Applied and Computational Mathematics*. Vol. 10, No. 6, 2021, pp. 138-145.
doi: 10.11648/j.acm.20211006.12

Received: September 14, 2021; **Accepted:** October 11, 2021; **Published:** November 11, 2021

Abstract: This document was an opportunity for us to measure the contributions of researchers on the asymptotic behavior of the extremes random variables. Beyond the available results, we have proposed an analysis of the behavior of the extremes of random variables of geometric type. We succeeded in determining a subsequence which allows us to establish a convergence in law of the extremes of this type of random variable while passing by the determination of a speed of convergence. We then exposed the limited law which results from it then we called upon the copulas of the extreme values to propose a joint limited law for two independent samples of random variables of geometric type. These results will allow us to analyze, in a document, not only the convergence in moment of order of the other extremes of the random variables of geometric type but also the general asymptotic behavior of the extremes of a serie of random variables with integer value. This document was an opportunity for us to measure the contributions of researchers on the asymptotic behavior of the extremes random variables. Beyond the available results, we have proposed an analysis of the behavior of the extremes of random variables of geometric type. We first made the case of the fact that the random variables of geometric type could be constructed from the random variables of exponential distribution and that they were not only integer variables but also that in general there were no sequences standards that allowed their extremes to converge. To do this, we first built a convergent $\phi(k)$ subsequence which we then used to define a geometric type $T_\phi(k)$ subsequence of random variables. We have also proved the convergence in distribution of the extremes of the random variables $T_\phi(k)$. We have also exhibited the resulting limit law. Finally, in this document, we have dealt with the multivariate case of random variables of geometric type. We considered two independent samples of random variables of geometric types. Using a copula of extreme values, in particular the logistic copula, we proposed a joint limit distribution of two independent samples of subsequences of geometric type random variables. We then exposed the limited law which results from it then we called upon the copulas of the extreme values to propose a joint limited law for two independent samples of random variables of geometric type.

Keywords: Asymptotic Convergence, Generalized Extreme Value Distribution, Exponantial and Geometric Distribution, Extreme Values Copulas

1. Introduction

One of the objectives of actuaries is to model the occurrence of rare events such as famine, wars, floods etc. Most often, a family of continuous probability laws is used to model extreme value phenomena. Let $(X_n)_{n\geq 0}$ be a sequence of independent and identically distributed random variables with common distribution function F. Let $\overline{X}_n$ is the unilateral maximum of a sequence $(X_n)_{n\geq 1}$ of real valued random variables (r.v). Gnedenko's Theorem teaches us that there are sequences of strictly positive reals $\{a_n\}_{n\leq 1}$ and reals $\{b_n\}$ such that $\dfrac{\overline{X}_n - b_n}{a_n}$ converges in law towards a limit law, then the only possible limit laws, defined by their non degenerate distribution function G. These are Generalized Extreme Values laws.

The work of Z. Peng [13] and [7] are interested in the convergence rate of the moments of extreme. As for C.W. Anderson [4] and F. Thomas and al [5], their work focused on the behavior of extremes for classes of discrete random variables. [9] focus on Limiting forms of frequency distribution of the largest or smallest member of a sample. As for [10], they worked on extreme values and their links with regularly varying functions.

For our part, we will focus, in this document, on the asymptotic behavior of the extremes of random variables of geometric type. These variables can be constructed from random variables of the exponential type. Define the random variable $\overline{T}_n = \max\{T_1, ..., T_n\}$.

$\overline{T}_n$ is an integer random variable and $(T_n)_{n\geq 1}$ is a sequence of i.i.d random variable with same geometric law.

From J. Galambos [6] there are no standardization suites a_n and b_n such that $\dfrac{\overline{T}_n - b_n}{a_n}$ converges in law when n tends to infinity. Our aim in this document is to exhibit a_n and b_n then a integer values subsequence $\phi(k)$ in order to analyze the convergence of $\dfrac{\overline{T}_{\phi(k)} - b_{\phi(k)}}{a_{\phi(k)}}$.

Subsequently, in this document, having the limit distribution of a sequence of r.v of laws of geometric type, we will use the copulas of the extreme values for a multivariate modeling of the law of extremes of a certain number of random variables.

Let $(X_1, ..., X_n)$ a sequence of random variables whose distribution function $H(x_1, ..., x_n)$ is a law of multidimensional extreme values. If $F_1,, F_n$ are the marginal laws of $X_1, ..., X_n$, we have

$$H(x_1, ..., x_n) = C\Big(F_l(x_1), ..., F_n(x_n)\Big)$$

where C is a copula of multidimensional extreme values.

Using C as a logistical copula, we offer a joint limited law for two independent samples of random variables of geometric type.

2. Preliminary

Let $(X_n)_{n\geq 0}$ be a sequence of independent and identically distributed random variables with common distribution function F. Let $\overline{X}_n$ is the unilateral maximum of a sequence $(X_n)_{n\geq 1}$ of real valued random variables (r.v)

$$\overline{X}_n := max\{X_1, \cdots, X_n\}, \quad n \geq 1 \tag{1}$$

Let again ϕ be the cumulative distribution function (cdf) of a real r.v and denote ϕ^{-1} the right inverse of ϕ

$$\phi^{-1}(t) := \inf\{s \in \mathbb{R},\ \phi(s) > t\}, \quad t \in]0, 1[\tag{2}$$

Let's start by recalling the definition of the laws of Gumbel, Frechet and Weibull.

Definition 2.1. Let G be a non-degenerate distribution function given by

$$\begin{cases} G_\gamma(x) = exp\{-(1+\gamma x)^{-\frac{1}{\gamma}}\} & , 1+\gamma x \geq 0 \text{ and } \gamma \neq 0 \\ G_0(x) = exp\{-exp(-x)\} \end{cases} \tag{3}$$

Explicitly the three laws are given by:

$$\begin{aligned} G(x) &= \Lambda(x) = e^{-e^{-x}}, \quad x \in \mathbb{R} \text{ (Gumbel law)} \\ G(x) &= e^{-x^{-\beta}}, \quad x \geq 0, \beta > 0 \text{ (Frechet law)} \\ G(x) &= e^{-(-x)\beta}, \quad x \leq 0, \beta > 0 \text{ (Weibull law)} \end{aligned}$$

The following lemma states the Fisher-Tippet's theorem which establishes the conditions of convergence towards the above limit laws.

Lemma 2.1. If there are sequences $a_n > 0$, $b_n \in \mathbb{R}$ and a non-degenerate distribution function H such that

$$\lim_{n\longrightarrow\infty} P\Big(\frac{\overline{X}_n - b_n}{a_n} \leq x\Big) = H(x), \qquad x \in \mathbb{R}$$

then H belongs to one of the families of the three types Gumbel, Frechet or Weibull laws.

For the proof of lemma (2.1), see [6].

In the literature,see [1, 2], it is proved that if $(X_n)_{n\geq 1}$ is a sequence of independent and identically distributed v.a then we can find normalization sequences a_n and b_n such that the law of $\frac{\overline{X}_n - a_n}{b_n}$ converges to Gumbel's law. We propose in this document to analyze the asymptotic convergence of the extremes of a sequence of v.a i.i.d of geometric law. Define the random variable $\overline{T}_n = \max\{T_1, ..., T_n\}$.

$\overline{T}_n$ is an integer random variable. $(T_n)_{n\geq 1}$ is a sequence of i.i.d random variable with same geometric law.

Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d random variable with same exponential law, the following lemma allows to find b_n et a_n which applies the Fisher-Tippet theorem.

Lemma 2.2.
Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d random variable with same exponential law and let $\overline{X}_n = \max_{i=1,...n}\{X_i\}$. There are real sequences a_n et $b_n > 0$ such us $\frac{\overline{X}_n - a_n}{b_n}$ converges in law to a random variable Z of Gumbel.

Proof of Lemma (2.2)

Suppose X_1,..., X_n be an i.i.d sequence with common exponential distribution $\mathcal{E}(1)$. Let $\overline{X}_n = \max_{i=1,...n}\{X_i\}$.

Consider reals sequences a_n and $b_n > 0$

$$\begin{aligned} p &= P(\frac{\overline{X}_n - a_n}{b_n} \leq x) \\ &= P(\max_{i=1,\dots n}\{X_i\} \leq xb_n + a_n) \\ &= P(X_1 \leq xb_n + a_n, ..., X_n \leq xb_n + a_n) \\ &= \prod_{i=1}^{n} P(X_i \leq xb_n + a_n) \\ &= \Big(P(X \leq xb_n + a_n)\Big)^n \end{aligned}$$

but $P(X \leq x) = 1 - e^{-x}$. It's comes that

$$P(\frac{\overline{X}_n - a_n}{b_n} \leq x) = \Big(1 - e^{-xb_n - a_n}\Big)^n = \Big(1 - \frac{e^{-xb_n}}{e^{a_n}}\Big)^n$$

This sequence converges on one of the laws, Gumbel, Frechet, Weibul. For $a_n = l(n)$ et $b_n = 1$, on a

$$\begin{aligned} \lim_{n\longrightarrow+\infty} P(\frac{\overline{X}_n - a_n}{b_n} \leq x) &= \lim_{n\longrightarrow+\infty}\Big(1 - \frac{e^{-xb_n}}{e^{a_n}}\Big)^n \\ &= e^{e^{-x}} \end{aligned}$$

then $\frac{\overline{X}_n - a_n}{b_n}$ converges to Gumbel's law.

Suppose that $(T_n)_{n\geq 1}$ will stand for a sequence of i.i.d. r.v.'s valued in $\{1, 2, \cdots\}$ whose law satisfies (8), i.e.

$$\overline{G}(k) = P(T_1 > k) = a\lambda^k(1 + \varepsilon_k), \quad \forall k \geq 1 \tag{4}$$

where G is the cdf of T_1, $a > 0$, λ belongs to $]0, 1[$ and $\lim_{k\to\infty} \varepsilon_k = 0$. It is convenient to set

$$\rho := -\ln\lambda. \tag{5}$$

We wish to focus on the asymptotic law of extremes of a geometric law. Thus the following sequence of real numbers in $[0, 1]$ will play an important role

$$u_n := \frac{\ln n}{\rho} - \Big\lfloor\frac{\ln n}{\rho}\Big\rfloor, \quad n \geq 1. \tag{6}$$

$\lfloor x\rfloor = \lceil x\rceil - 1$ is the integer part of x.

We will exhibit real sequences a_n and b_n then a integer values subsequences $\phi(k)$ in order to analyze the convergence of $u_\phi(k)$ and appreciate the convergence in law of $\frac{\overline{T}_{\phi(k)} - b_{\phi(k)}}{a_{\phi(k)}}$ towards a limit distribution that we will determine .

Subsequently having the limit distribution of a sequence of r.v of laws of geometric type, we will use the copulas of the extreme values for a multivariate modeling of the law of extremes of a certain number of random variables.

Definition 2.2. We call n-dimensional copula, any multivariate distribution function C having for marginal the uniform distribution on $[0; 1]$.

Let $(X_1, ..., X_n)$ a tuple of random variables whose distribution function $H(x_1, ..., x_n)$ is a law of multidimensional extreme values. If F_1,,, F_n are the marginal laws of $X_1, ..., X_n$, we have $H(x_1, ..., x_n) = C\Big(F_l(x_1), ..., F_n(x_n)\Big)$, where C is a copula of multidimensional extreme values.

Authors in [14, 15, 16] have deduced the general shape of multivariate extreme value copulas as follows:

$$C(u_1, ..., u_n) = exp\Big\{(\sum_{i=1}^{n}\log(u_i))A\Big(\frac{\log(u_1)}{\sum_{i=1}^{n}\log(u_i)}\Big)\Big\} \tag{7}$$

$u_i \in]0, 1[\ \forall i \in \{0, ..., n\}$. A is a convex function defined on $[0, 1]$. The function A has a important role in the study of the extremal behavior of a pair of (U, V) with joint distribution C. It expresses the natural tendency of v.a U and V to take large values simultaneously. A good estimate of A is quite critical for its use. Authors like [8] and [3] tell us more about this topic.

3. Main Results

Theorem 3.1. Let $(T_n)_{n\geq 1}$ be a sequence of iid rv with values in $\{1, 2, ...\}$ and with the same pseudo-geometric distribution given by

$$P(T_1 > k) = a\lambda^k(1 + \epsilon_k) \tag{8}$$

where $a > 0$, $\lambda \in]0, 1[$ and $\lim_{k\longrightarrow\infty} \epsilon_k = 0$. Thus

1. We define for $u \in [0, 1]$

$$(\phi_k)_{k\geq 1} = \Big(\Big\lfloor exp\{(k + u)\rho\}\Big\rfloor + 1\Big)_{k\geq 1} \tag{9}$$

Then,

$$\overline{T}_{\phi_k} - 1 - \Big\lfloor\frac{\ln(\phi_k)}{\rho}\Big\rfloor \quad \text{converges in law to}$$

$$\Big\lfloor u + \frac{Z + \ln(a)}{\rho}\Big\rfloor \quad \text{when } k \longrightarrow \infty \tag{10}$$

Z is rv with Gumbel's law.

2. Conversely, if there is a subsequence $\phi_k \uparrow \infty$ such us $\overline{T}_{\phi_k} - 1 - \Big\lfloor\frac{\ln(\phi_k)}{\rho}\Big\rfloor$ converges in distribution then we can find $u \in [0, 1]$ as we have (10).

Proof of Theorem 3.1 Let $(T_n)_{n\geq 1}$ is a sequence of i.i.d random variable with same geometric law.

We stated in the introduction that from J. Galambos there are no standardization sequences a_n and b_n such that $\frac{\overline{T}_n - b_n}{a_n}$ converges in law when n tends to infinity.

How we can analyze the asymptotic behavior of T_n $n \leq 1$. Let us first construct a sequence of random variable(r.v) of geometric type from a series of r.v of exponential law.

Lemma 3.1. Let $X_1, ..., X_n$ be a sequence of iid r.v with common exponential law $\mathcal{E}(1)$ then there are real constants a and b such us $T_i = \lfloor\frac{X_i - a}{b}\rfloor$ be an geometric r.v.

Moreover

$$P(T_i = k) = \lambda^{k-1}(1 - \lambda)$$

Proof of Lemma (3.1)

Consider $X_1, ..., X_n$ a sequence of iid rv with common exponential law $\mathcal{E}(1)$. Let a and b be real constants. let's suppose that $T_i = \lfloor \frac{X_i - a}{b} \rfloor$

$$\begin{aligned} P(T_i = k) &= P\Big(\lfloor \frac{X_i - a}{b} \rfloor = k\Big) \\ &= P\Big(\lfloor \frac{X_i}{b} \rfloor - \lfloor \frac{a}{b} \rfloor = k\Big) \\ &= P\Big(\lfloor \frac{X_i}{b} \rfloor = k + \lfloor \frac{a}{b} \rfloor\Big) \\ &= P\Big(k + \lfloor \frac{a}{b} \rfloor \le \frac{X_i}{b} < k + \lfloor \frac{a}{b} \rfloor + 1\Big) \\ &= P\Big(kb + \lfloor a \rfloor \le X_i < kb + \lfloor a \rfloor + b\Big) \\ &= 1 - e^{-kb - \lfloor a \rfloor - b} - 1 + e^{-kb - \lfloor a \rfloor} \\ &= e^{-kb - \lfloor a \rfloor} - e^{-kb - \lfloor a \rfloor - b} \\ P(T_i = k) &= e^{-kb}\Big(e^{-\lfloor a \rfloor} - e^{-\lfloor a \rfloor - b}\Big) \end{aligned}$$

Let $b = -\ln(\lambda)$ and $a = \ln(\lambda)$. Then

$$\begin{aligned} P(T_i = k) &= e^{k \ln(\lambda)}\Big(e^{-\lfloor \lambda \rfloor} - e^{-\lfloor \ln(\lambda) \rfloor + \ln(\lambda)}\Big) \\ P(T_i = k) &= \lambda^k(\frac{1}{\lambda} - 1) = \lambda^{k-1}(1 - \lambda) \end{aligned}$$

We deduce that T_i is a rv of geometric type. The lemma is proven. we can express: Let $(T_n)_{n \ge 1}$ be a sequence of i.i.d random variable with same geometric law. Then we have

$$\begin{aligned} \overline{T}_n &= \max_{1 \le i \le n} \{T_i\} = \max_{1 \le i \le n} \Big\{ \Big\lfloor \frac{X_i}{\rho} \Big\rfloor + 1 \Big\} \\ &= \Big\lfloor \max_{1 \le i \le n} \Big\{ \frac{X_i}{\rho} \Big\} \Big\rfloor + 1 = \Big\lfloor \frac{\overline{X}_n}{\rho} \Big\rfloor + 1 \end{aligned} \tag{11}$$

Consider the following lemma:

Lemma 3.2. For any $u \in [0, 1]$ there exists a subsequence $(\phi(k))$ such that $\lim_{k \to \infty} u_{\phi(k)} = u$.

Proof of Lemma(3.2) Let $X_1, ..., X_n$ be a sequence of i.i.d r.v with common exponantial law $\mathcal{E}(1)$. Let $\overline{X}_n = \max_{i=1,...n}\{X_i\}$. We have shown that $\overline{X}_n - \ln(n)$ converges in distribution to a Gumbel variable Z.

Let $T_i = \left\lfloor \frac{X_i}{\rho} \right\rfloor + 1$ a random variable of geometric type. Consider the following sequence

$$u_n = \frac{\ln(n)}{\rho} - \Big\lfloor \frac{\ln(n)}{\rho} \Big\rfloor.$$

It is clear that $u_n \in [0, 1]$. Let a subsequence $\phi(k)$ such as $\lim_{k \to \infty} u_{\phi(k)} = u$.

1. Let us first show that the set of adhesion values of the sequence $(u_n)_{n \ge 1}$ is $[0, 1]$. Let $0 \le \alpha < \beta \le 1$ such that $\alpha < u_n < \beta$. Let $k = \left\lfloor \frac{\ln(n)}{\rho} \right\rfloor$.

   We have $\alpha + k < \frac{\ln(n)}{\rho} < \beta + k \Longrightarrow e^{\rho(\alpha+k)} < n < e^{\rho(\beta+k)}$.

   A necessary condition for the existence of n is that:

   $$e^{\rho(\beta+k)} - e^{\rho(\alpha+k)} \ge 1. \tag{12}$$

   On this basis, there exists an integer dependent on k such that $\alpha < u_n < \beta$. Consider that this integer is denoted by $\phi(k)$.

2. Now let's build $\phi(k)$.

   The condition (12) implies that $k \le \frac{1}{\rho} \ln\left(\frac{1}{e^{\rho\beta} - e^{\rho\alpha}}\right)$.

   Let $q = \left\lfloor \frac{\ln(n)}{\rho} \right\rfloor + 1$ such as for all $k \le 2$, we have $k > q$.

   Suppose $\phi(k) = \left\lfloor e^{\rho(\alpha+k)} \right\rfloor$.

   We have successively :

   $$e^{(k+\alpha)\rho} \le \phi(k) < e^{(k+\alpha)\rho} + 1 < e^{(k+\beta)\rho}$$

   and

   $$k + \alpha \le \frac{\ln\big(\phi(k)\big)}{\rho} < k + \beta.$$

   Since $(\alpha, \beta) \ne (0, 1)$, the above inequality implies that

   $$\Big\lfloor \frac{\ln\big(\phi(k)\big)}{\rho} \Big\rfloor = k.$$

   As result

   $$u_{\phi(k)} = \frac{\ln\big(\phi(k)\big)}{\rho} - \Big\lfloor \frac{\ln\big(\phi(k)\big)}{\rho} \Big\rfloor$$

   belongs to $[\alpha, \beta]$.

   This proves that $\overline{\{u_n,\ n \ge 1\}} = [0, 1]$.

This is the proof of lemma(3.2).

So for all $u \in [0, 1]$, we deduce that for $\phi(k)$ thus as defined we have $\lim_{k \longrightarrow \infty} u_{\phi(k)} = u$. Consider following lemma of convergence with subsequence $(\phi(k))_{k \ge 1}$.

Lemma 3.3. Let $(T_n)_{n \ge 1}$ be a sequence of i.i.d random variable with same geometric law. Suppose that for all $u \in [0, 1]$, there is a subsequence $\phi(k)$ such as

$$\lim_{k \longrightarrow \infty} \frac{\ln\big(\phi(k)\big)}{\rho} - \Big\lfloor \frac{\ln\big(\phi(k)\big)}{\rho} \Big\rfloor = u.$$

Then

$$\overline{T}_{\phi(k)} - 1 - \Big\lfloor \frac{\ln(\phi(k))}{\rho} \Big\rfloor \quad \text{converges in law to}$$

$$\Big\lfloor u + \frac{Z + \ln(a)}{\rho} \Big\rfloor \quad \text{when } k \longrightarrow \infty$$

Proof of Lemma(3.3) Let $X_1, ..., X_n$ be a sequence of i.i.d r.v with common exponantial law $\mathcal{E}(1)$ and let $\overline{T}_n = \left\lfloor \frac{\overline{X}_n}{\rho} \right\rfloor + 1$ be is a sequence of i.i.d random variable with same geometric law. Let's introduce the sequence $u_n = \frac{\ln(n)}{\rho} - \left\lfloor \frac{\ln(n)}{\rho} \right\rfloor$, we have shown that for all $u \in [0, 1]$, there is a subsequence $\phi(k)$ such as $\lim_{k \longrightarrow \infty} u_{\phi(k)} = u$.

From the previous lemmas, we realized that:

If $\overline{X}_n = \max_{1\le i\le n}\{X_i\}$ then

$$\overline{X}_n - \ln(n) \quad \text{converges in law to Gumbel's law}$$

$$\Gamma(x) = e^{-e^{-x}}, \quad x \ge 0.$$

Let $Z_n = \overline{X}_n - \ln(n)$, it comes that

$$\overline{T}_{\phi(k)} = \Big\lfloor \frac{Z_{\phi(k)}}{\rho} + \frac{\ln(\phi(k))}{\rho} \Big\rfloor + 1$$

$$\Longrightarrow \overline{T}_{\phi(k)} = \Big\lfloor \frac{Z_{\phi(k)}}{\rho} + \Big\lfloor \frac{\ln(\phi(k))}{\rho} \Big\rfloor + u_{\phi(k)} \Big\rfloor + 1.$$

so

$$\overline{T}_{\phi(k)} = \Big\lfloor \frac{Z_{\phi(k)}}{\rho} + u_{\phi(k)} \Big\rfloor + \Big\lfloor \frac{\ln(\phi(k))}{\rho} \Big\rfloor + 1$$

$$\Longrightarrow \overline{T}_{\phi(k)} - \Big\lfloor \frac{\ln(\phi(k))}{\rho} \Big\rfloor - 1 = \Big\lfloor \frac{Z_{\phi(k)}}{\rho} + u_{\phi(k)} \Big\rfloor.$$

Moreover

$$\frac{Z_{\phi(k)}}{\rho} + u_{\phi(k)} \quad \text{converges in law to } \tfrac{Z}{\rho} + u.$$

So it comes that:

$$\overline{T}_{\phi(k)} - \Big\lfloor \frac{\ln(\phi(k))}{\rho} \Big\rfloor - 1 \quad \text{converge in law to } \Big\lfloor \tfrac{Z}{\rho} + u \Big\rfloor$$

End of proof. Let us now prove the converse. Consider first the lemma

Lemma 3.4.
Let T be a $\{1, 2, \cdots\}$-valued r.v. with cumulative distribution function ϕ. Then T is distributed as $\phi^{-1}\big(1 - e^{-X}\big)$ where X is a r.v. with exponential distribution.

The proof of Lemma 3.4 is classical since the law of $1-e^{-X}$ is uniform over $[0, 1]$. Let's use the following general result:

Lemma 3.5. Consider a real valued sequence $(y_k)_{k\ge1}$ converging at infinity to a real number y. We suppose that y does not belong to $\{p\rho - u\rho - \ln(a), p \in \mathbb{Z}\}$, then for large k

$$F^{-1}\Big(1-\frac{1}{\phi(k)}e^{-y_k}\Big)-1-\Big\lfloor \frac{\ln(\phi(k))}{\rho} \Big\rfloor = \Big\lfloor u+\frac{1}{\rho}\{y+\ln a\} \Big\rfloor. \tag{13}$$

The proof of this lemma is given by

Proof of Lemma (3.5)

It is clear that $\lim_{k\to\infty} \dfrac{1}{\phi(k)} e^{-y_k} = 0$. Therefore

$$m_k := F^{-1}\Big(1 - \frac{1}{\phi(k)} e^{-y_k}\Big)$$

goes to infinity as $k \to \infty$ and

$$\lim_{k\to\infty} \ln(1 + \varepsilon_{m_k}) = \lim_{k\to\infty} \ln(1 + \varepsilon_{m_k-1}) = 0.$$

Using moreover the definition of u_n (cf (6)) we deduce :

$$\lim_{k\to\infty} m'_k = \lim_{k\to\infty} m''_k = u + \frac{1}{\rho}\{y + \ln a\}$$

where

$$\begin{aligned} m'_k &= \frac{1}{\rho}\big\{ \ln(\phi(k)) + y_k + \ln a + \ln(1 + \varepsilon_{m_k})\big\} \\ &- \Big\lfloor \frac{\ln(\phi(k))}{\rho} \Big\rfloor \end{aligned}$$

$$\begin{aligned} m''_k &= \frac{1}{\rho}\big\{ \ln(\phi(k)) + y_k + \ln a + \ln(1 + \varepsilon_{m_k-1})\big\} \\ &- \Big\lfloor \frac{\ln(\phi(k))}{\rho} \Big\rfloor \end{aligned}$$

Using Lemma (3.2) we get

$$m'_k - 1 < m_k - 1 - \Big\lfloor \frac{\ln(\phi(k))}{\rho} \Big\rfloor \le m''_k.$$

Our assumption over y implies that $x := u + \dfrac{1}{\rho}\{y + \ln a\}$ is not an integer. Therefore the only one integer in the interval $[x - 1, x]$ is $\lfloor x \rfloor$.

The result follows immediately.

This lemma is proven.

Proof of Lemma (3.4) In this lemma, we only deal with the distribution of $(T_n)_{n\ge1}$ we can suppose that

$$T_n = G^{-1}\big(1 - e^{-X_n}\big), \quad n \ge 1. \tag{14}$$

The function G^{-1} being non-decreasing we deduce :

$$\overline{T}_n = G^{-1}\big(1 - e^{-\overline{X}_n}\big), \quad n \ge 1.$$

From Resnick [11] $\overline{X}_n - \ln n$ converges in distribution to Z as n goes to infinity . It is clear that the above identity can be written in the following form :

$$\overline{T}_n = G^{-1}\Big(1 - \frac{1}{n} e^{-(\overline{X}_n - \ln n)}\Big).$$

From the Skorokhod theorem [1], we can find a sequence $(W_n)_{n\ge1}$ such that for any n, the r.v. $\overline{X}_n - \ln n$ is distributed as W_n and $(W_n)_{n\ge1}$ converges almost surely to a r.v. W. Obviously W and Z have the same law. Therefore, for any n, $\overline{T}_n$ is distributed as $G^{-1}\Big(1 - \frac{1}{n} e^{-W_n}\Big)$. With any lost of generality we can suppose :

$$\overline{T}_n = G^{-1}\Big(1 - \frac{1}{n} e^{-W_n}\Big), \quad n \ge 1. \tag{15}$$

The above identity (15) is the key of the proof of Theorem 3.1.

We suppose there exists a subsequence $(\phi(k))_{k\ge1}$ such that:

$$\overline{T}_{\phi(k)} - 1 - \Big\lfloor \frac{\ln(\phi(k))}{\rho} \Big\rfloor \quad \text{converges in law as } k \to \infty. \tag{16}$$

We claim that $v_k := u_{\phi(k)} = \frac{\ln(\phi(k))}{\rho} - \left\lfloor \frac{\ln(\phi(k))}{\rho} \right\rfloor$ is a convergent sequence. Since v_k belongs to $[0,1]$, it is equivalent to show that if $\phi, \psi : \mathbb{N} \to \mathbb{N}$ are two increasing functions such that $\lim_{i\to\infty} v_{\phi(i)} = u$ and $\lim_{i\to\infty} v_{\psi(i)} = u'$, then $u = u'$.

We can note that

$$\overline{T}_{\phi(i)} - 1 - \left\lfloor \frac{\ln(\phi(i))}{\rho} \right\rfloor \quad \text{converges in distribution to}$$

$$\left\lfloor u + \frac{Z + \ln a}{\rho} \right\rfloor \quad \text{as } i \to \infty$$

and

$$\overline{T}_{\psi(i)} - 1 - \left\lfloor \frac{\ln(\psi(i))}{\rho} \right\rfloor \quad \text{converges in distribution to}$$

$$\left\lfloor u' + \frac{Z + \ln a}{\rho} \right\rfloor \quad \text{as } i \to \infty$$

Therefore, $\left\lfloor u + \frac{Z+\ln a}{\rho} \right\rfloor$ and $\left\lfloor u' + \frac{Z+\ln a}{\rho} \right\rfloor$ have the same distribution. The result follows using following lemma with $\xi := u \wedge u' + \frac{Z+\ln a}{\rho}$ and $\tau := |u - u'|$.

These results allow us to ensure the convergence in law of the extremes of r.v of geometric type.

End of proof.

In what will follow, we will analyze the convergence in law of extremes of r.v $\overline{T}_{\phi(k)}$ of geometric type according to the exhibited subsequences. We arrive at the result.

Consider $(X_1, ..., X_n)$ et $(Y_1, ..., Y_m)$ independent samples of r.v. $\overline{T}_n = \left\lfloor \frac{\overline{X}_n}{\rho_1} \right\rfloor + 1$ et $\overline{T'}_m = \left\lfloor \frac{\overline{Y}_m}{\rho_2} \right\rfloor + 1$

Theorem 3.2.

Let $\overline{T}_n = \left\lfloor \frac{\overline{X}_n}{\rho_1} \right\rfloor + 1$ et $\overline{T'}_m = \left\lfloor \frac{\overline{Y}_m}{\rho_2} \right\rfloor + 1$ and if there is a_1, a_2, b_1, b_2 real numbers such as $\frac{\overline{T}_{\phi(k)} - a_1}{b_1}$ et $\frac{\overline{T'}_{\phi(k)} - a_2}{b_2}$ converges respectively to T (of distribution F) and T'(of distribution G) then the joint distribution of $\Big(T, T'\Big)$ through the logistic copula is given by

$$\begin{aligned} H_\theta(x,y) &= exp\Big\{ - \Big(exp\{-\rho_1\theta(\lfloor x \rfloor - u + 1)\} \\ &+ exp\{-\rho_2\theta(\lfloor y \rfloor - v + 1)\}\Big)\Big\} \end{aligned} \tag{17}$$

Proof of Theorem (3.2) Consider first the following lemma

Lemma 3.6. Let $\phi(k)$ previously defined as

1. $u_{\phi(k)}$ converges to u when $k \longrightarrow +\infty$ and
2. $T_{\phi(k)}$ converges in law to T

Then

$$\sup_{x\in\mathbb{R}} \Big| F_{\phi(k)}(x) - F(x) \Big| \leq \frac{2e^{-2}+1}{\phi(k)-1}. \tag{18}$$

Proof of Lemma (3.6) Suppose k is large enough. We have $\frac{\ln \phi(k)}{\rho} \notin \mathbb{N}$ and so

$$\begin{aligned} F_{\phi(k)}(x) - F(x) &= \Lambda_{\phi(k)}\Big(\rho(\lfloor x \rfloor - u_{\phi(k)} + 1)\Big) \\ &- \Lambda\Big(\rho(\lfloor x \rfloor - u + 1)\Big) \end{aligned}$$

$$\begin{aligned} \Big| F_{\phi(k)}(x) - F(x) \Big| &\leq \Big| \Lambda_{\phi(k)}\Big(\rho(\lfloor x \rfloor - u_{\phi(k)} + 1)\Big) \\ &- \Lambda\Big(\rho(\lfloor x \rfloor - u_{\phi(k)} + 1)\Big)\Big| \\ &+ \Big| \Lambda\Big(\rho(\lfloor x \rfloor - u_{\phi(k)} + 1)\Big) \\ &- \Lambda\Big(\rho(\lfloor x \rfloor - u + 1)\Big)\Big| \end{aligned}$$

Thus:

$$0 \leq \rho(\lfloor x \rfloor - u + 1) - \rho(\lfloor x \rfloor - u_{\phi(k)} + 1) < exp\{-\rho(k+u)\}$$

Let $w_n(t) = \lfloor x \rfloor - u_n + 1$ and $w(t) = \lfloor x \rfloor - u + 1$. We have:

$$\begin{aligned} \Gamma_{\phi(k)} &= \Big| \Lambda\Big(\rho w_{\phi(k)}(x)\Big) - \Lambda\Big(\rho w(x)\Big)\Big| \\ &\leq \sup_{\rho w(x) < z < \rho w_{\phi(k)}(x)} \Lambda'(z)\Big(\rho w(x)) - \rho w_{\phi(k)}(c)\Big) \\ &\leq \sup_{\rho\lfloor x \rfloor < z < \rho\lceil x \rceil} \Lambda'(z) exp\{-\rho(k+u)\} \\ &= \Big(\Lambda'(\rho\lfloor x \rfloor)1_{\{x\geq 0\}} + \Lambda'(\rho\lceil x \rceil)1_{\{x<0\}}\Big) \\ &\times exp\{-\rho(k+u)\} \\ &\leq \frac{e^{\rho\lfloor x \rfloor}1_{\{x\geq 0\}} + e^{-\rho\lceil x \rceil - e^{-\rho\lceil x \rceil}}1_{\{x<0\}}}{\phi(k)-1} \\ &\leq \frac{1}{\phi(k)-1}. \end{aligned}$$

Consider again that

$$\begin{aligned} \Gamma_n &= \sup_{x\in\mathbb{R}} \Big| \Lambda_n(x) - \Lambda(x) \Big| \\ &= \sup_{x\in\mathbb{R}} \Big| \Big(1 - \frac{e^{-x}}{n}\Big)^n 1_{[-\ln n, \infty[}(x) - e^{-e^{-x}} \Big| \\ &= \sup_{y\geq 0} \Big| \Big(1 - \frac{y}{n}\Big)^n 1_{[0,n]}(y) - e^{-y} \Big| \\ &\leq (2 + \frac{1}{n})e^{-2}n^{-1} \end{aligned}$$

According to Hall and Wellner [12], by combining the two inequalities, we obtain

$$\begin{aligned} sup_{x\in\mathbb{R}} \Big| F_{\phi(k)}(x) - F(x) \Big| &\leq \frac{1}{\phi(k)-1} + \frac{(2+\frac{1}{\phi(k)})e^{-2}}{\phi(k)} \\ &\leq \frac{2e^{-2}+1}{\phi(k)-1}. \end{aligned} \tag{19}$$

This is the proof of lemma (3.6).

Consider

Lemma 3.7. Let F and G be distribution functions given respectively by

$$F(x) = \Lambda[\rho_1(\lfloor x \rfloor - u + 1)]$$

and

$$G(x) = \Lambda[\rho_2(\lfloor y \rfloor - v + 1)]$$

Let C_θ be bivariate copula of the extremes with the pickand dependency function

$$A_\theta(t) = \Big[t^\theta + (1-t)^\theta\Big]^{\frac{1}{\theta}} (\theta > 1)$$

Then

$$\begin{aligned} C_\theta(F(x), G(y)) &= exp\Big\{ - \Big(exp\{-\rho_1 \theta(\lfloor x \rfloor - u + 1)\} \\ &+ exp\{-\rho_2 \theta(\lfloor y \rfloor - v + 1)\}\Big)\Big\} \end{aligned}$$

Proof of Lemma (3.7)

Let F and G be distributions functions of r.v given by

$$F(x) = \Lambda[\rho_1(\lfloor x \rfloor - u + 1)]$$

and

$$G(x) = \Lambda[\rho_2(\lfloor y \rfloor - v + 1)]$$

Define the bivariate logistics copula by

$$C_\theta(u, v) = exp\Big\{\Big(\log(u) + \log(v)\Big) A\Big(\frac{\log(u)}{\log(u) + \log(v)}\Big)\Big\};$$

$$A_\theta(t) = \Big[t^\theta + (1-t)^\theta\Big]^{\frac{1}{\theta}} \quad (\theta > 1)$$

This copula allows us to measure the dependence of the rarest events.

$$\begin{aligned} C_\theta(u, v) &= exp\Big\{\Big(\log(u) + \log(v)\Big) \\ &\times \Big[\Big(\frac{\log(u)}{\log(u) + \log(v)}\Big)^\theta \\ &+ \Big(1 - \frac{\log(u)}{\log(u) + \log(v)}\Big)^\theta\Big]^{\frac{1}{\theta}}\Big\} \\ &= exp\Big\{\Big(\log(u) + \log(v)\Big) \\ &\times \Big[\Big(\frac{\log(u)}{\log(u) + \log(v)}\Big)^\theta \\ &+ \Big(\frac{\log(v)}{\log(u) + \log(v)}\Big)^\theta\Big]^{\frac{1}{\theta}}\Big\} \\ &= exp\Big\{\Big([\log(u)]^\theta + [\log(v)]^\theta\Big)^{\frac{1}{\theta}}\Big\} \end{aligned}$$

With reference to the F and G margins defined by

$$\begin{aligned} F(x) &= \Lambda[\rho_1(\lfloor x \rfloor - u + 1)] \\ &= exp\Big(- exp\{-\rho_1[\lfloor x \rfloor - u + 1]\}\Big) \end{aligned}$$

and

$$\begin{aligned} G(x) &= \Lambda[\rho_2(\lfloor y \rfloor - v + 1)] \\ &= exp\Big(exp\{-\rho_2[\lfloor y \rfloor - v + 1]\}\Big) \end{aligned}$$

It comes that

$$\begin{aligned} C_\theta(F(x), G(y)) &= exp\Big\{ - \Big(exp\{-\rho_1 \theta(\lfloor x \rfloor - u + 1)\} \\ &+ exp\{-\rho_2 \theta(\lfloor y \rfloor - v + 1)\}\Big)\Big\} \end{aligned}$$

End of proof.

Lemmas (3.6) and (3.7) complete the proof of the theorem (3.2).

4. Conclusion

In this article, we have analyzed the asymptotic behavior of the extremes of a sequence of independent and identically distributed random variables according to a geometric type law. For this purpose, we have constructed a convergent subsequence and this has enabled us to exhibit a subsequence of geometrical and independent random variables which converges in distribution. Finally, we have examined, through the copula of extreme value, the joint limit law of two independent samples of random variables of geometric type. These results will allow us to analyze, in a another document, not only the convergence in moment of order of the other extremes of the random variables of geometric type but also the general asymptotic behavior of the extremes of a series of random variables with integer value .

References

[1] A. V. Skorokhod *Limit Theorems for Stochastic Processes* Theory Probab. Appl., 1956, Vol. 1, No. 3: pp. 261-290, Society for Industrial and Applied Mathematics

[2] A. V. Stepanov *Limits theorem for weak records*, 1990.

[3] C. Genest, J. Segers, *Rank-based inference for bivariate extreme-value copulas* , Ann. Stat. 37 (2009), No 5B, p. 2990-3022.

[4] C. W. Anderson *Extreme value theory for a class of discrete distributions with applications to some stochastique processe*, J.Appl. Pron. No 7, pp 99-113, 1970.

[5] F. Thomas Bruss, Rudolf Grübel, *On the multiplicity of the maximum in a discrete random sample*, The annals of Applied Probability, vol.13, No 4, pp 1252-1263, 2003.

[6] J. Galambos, *The asymptotic theory of extreme order statistics*, juin 1978, pp 85.

[7] J. Pickands, *Moment convergence of sample extremes*, Ann. Math. Statist. 39 (1968), 881-889.

[8] K. Ghoudi, A. Khoudraji, L.-P. Rivest, *Propriétés statistiques des copules de valeurs extrêmes bidimensionnelles*, Can. J. Stat. 26 (1998), No 1, p. 187-197.

[9] R. A. Fisher et L. H. C. Tippet, *Limiting forms of frequency distribution of the largest or smallest member of a sample*, Mathematical Proceedings of the Cambridge Philosophical Society, volume 24, numéro 2, Acril 1928, pp. 180-190.

[10] S. I. Resnick, *Extreme values, Regular variation and Point processes*, Springer-Verlag, New york, 1987, pp. 443-45.

[11] T. Bastogne, R.Keinj and P.Vallois *Multinomial model-based formulations of tcp and ntcp for radiotherapy treatment planning*. Journal of Theoretical Biology, 279: 55-62, 2011.

[12] W. J. Hall, Jon A. Wellner *Estimation of Mean Residual Life*, Statistical Modeling for Biological Systems pp 169-189, jul 2017.

[13] Z. Peng and S. Nadarajah *Convergence rate for the moments of extremes*, Bull. Korean Math. Soc. 49(2012) No. 3, pp. 495-510.

[14] Tiago de Olivera, *Extremal distributions* Rev. Fac. Ciencas Univ. Lisboa, A8, 299-310, 1958.

[15] Sibuya, M. Bivariate extreme statistics Ann. Inst. Statist. Math., 11, 195-210, 1960.

[16] Geoffroy, J. Contribution B la thtorie des valeurs extremes Publ. Inst. Statist. Univ. Paris, 7, 37-185, 1958.