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Performance measure of binomial model for pricing American and European options

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Abstract: Binomial model is a powerful technique that can be used to solve many complex option-pricing problems. In contrast to the Black-Scholes model and other option pricing models that require solutions to stochastic differential equations, the binomial option pricing model is mathematically simple. It is based on the assumption of no arbitrage. The assumption of no arbitrage implies that all risk-free investments earn the risk-free rate of return and no investment opportunities exists that requires zero amount of investment but yield positive returns. It is the activity of many individuals operating within the context of financial market that, in fact, upholds these conditions. The activities of the arbitrageurs or speculators are often maligned in the media, but their activities insure that financial markets work. They insure that financial assets such as options are priced within a narrow tolerance of their theoretical values. In this paper we use binomial model to derive the Black-Scholes equation using the risk-neutral expectation formula. We also use binomial model for the valuation of European and American options. Lastly, the primary reason why the binomial model is used is its flexibility compared to the Black-Scholes model and it is also used to price a wide variety of options.

Keywords: American Option, Black-Scholes Model, Binomial Model, European Option

1. Introduction

Options have been considered to be the most dynamic segment of the security markets since the inception of the Chicago Board Options Exchange (CBOE) in April, 1973 with more than one million contacts per day, CBOE is the largest business option exchange in the world. After that, several other option exchanges such as London International Financial Futures and Options Exchange (LIFE EURONEXT) had been set up.

Option is a major financial derivative, it gives the holder of that options the right, but not the obligation to trade a fixed amount of underlying asset at an agreed-upon price on the maturity date (European option) or anytime on or before the maturity date (American option). A call option gives the holder the right but not the obligation to buy; a put option gives the holder the right but not the obligation to sell.

Over the last a few decades, due to the famous Black and Scholes work, option valuation has gained a lot of attention. In Black and Scholes [2] seminar paper, the assumption of log-normality was obtained and its application for valuing various ranges of financial instruments and derivatives is considered essential.

Options form the foundation of innovative financial instruments which are extremely versatile securities that can be used in different ways. Over the past decade, option has been developed to provide the basis for corporate hedging and for the asset/liability management of financial institutions.

Option pricing theory has a long history, but it was not until Black and Scholes presented the first complete equilibrium option pricing model in the year 1973. Moreover, in the same year, Robert Merton extended the Black-Scholes (BS) model in several important ways. Since its invention, the BS model has been widely used by traders to determine the price for an option. However, this famous formula has been questioned after the 1987 crash.

Following the Black-Scholes option pricing model in 1973, a number of other popular approaches was developed such as binomial tree model by Cox-Ross-Rubinstein [9], Jump diffusion model suggested by Merton [17], the Monte

Carlo simulation method developed by Boyle [6] and finite difference method to price the derivative governed by solving the underlying partial differential equation by Brennan and Schwartz [7] just to mention a few.

The complexity of option pricing formulas and the demand of speed in financial trading market require fast ways to process these calculations; as a result of this the development of computational methods for option pricing models can be the only solution.

Even in the 70’s Black-Scholes calculator is a must for option traders. As well as the option market, computing the industry developed dramatically since 1970s. Computer calculation speed is getting faster and faster today. Speculate option traders are using a selection of software applications to run the option pricing models to price the derivative, then compare the market price to looking for the mispricing opportunity to invest and act quickly in order to make a profit.

A considerable amount of empirical evidence in literature suggesting that the Black-Scholes which assumes asset returns follow continuous diffusion process with constant conditional volatility is inconsistent with the statistical properties of many asset prices. Implied volatilities calculated by reverting Black-Scholes model are higher for deep in-the-money and out-of-the-money options, the existence of volatility smile indicates the mispricing problem.

Many researchers have challenged the validity of Black-Scholes model using the empirical tests which were based on the historical data set, this motivated us to produce an application to apply the real-time market quote data to apply for the Black-Scholes model, and then compare the output price with the market option price quote at that time point.

For exhaustive or relevant literatures on option pricing see [1], [3], [4], [5], [8], [10], [11], [12], [13], [14], [15], [16], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27] and [28].

1.1. Definition of Some Basic Terms

This section presents some definition of terms as follows:

1.1.1. Financial Derivative

A financial derivative also known as a derivative security can be defined as a financial instrument whose value depend on (or is derived from) the value of other more basic underlying variables or security.

1.1.2. Options

It can be defined as a financial contract or derivative that represents a contract sold by one party (option writer) to another party (option holder). The contract offer the buyer the right but not the obligation to buy (call) or sell (put) a security or other financial asset at an agreed upon price (the strike price) during a certain period of time.

1.1.3. Call Option

This gives the holder of the option the right to buy the underlying asset or security at a specified price for a certain fixed period of time. Call option gives the option to buy at a certain price and so the buyer would want the stock to go up. The European call option payoff can be expressed as:

$$C_E(S_T, T) = (S_T - K)+$$

$$where\ T = Time\ of\ expiry$$

$$K = strike\ price$$

1.1.4. Put Option

This gives the holder of that option the right to sell the underlying asset or security at a specified price for a certain fixed period of time. This gives the option to sell at a certain price so the buyer would want the stock to go down. The European put payoff $\mathrm{P}(S_T)$ can be expressed as

$$P_E(S_T, T) = (K - S_T)+$$

$$where\ T = Time\ of\ expiry$$

$$K = Strike\ price$$

1.1.5. At-The-Money

An option is at-the-money, if the strike price of the option is equal to the market price of the underlying security.

i.e. if $S = K$, then option is at the money

1.1.6. In-The-Money

A call option is in-the-money if the strike price is less than the market price of the underlying security. A put option is said to be in-the-money if the strike price is greater than the market price of the underlying security.

1.1.7. Out-of-The-Money

A call option is out-of-the-money if the strike price is greater than the market price of the underlying security. A put option is out-of-the-money if the strike price is less than market price of the underlying security.

1.1.8. Hedgers

Hedge can be defined as a conservative strategy used to limit investment loss by effecting a transaction which offsets an existing position. Hedgers are risk-reducers and as hedge is a trade designed to reduce risks.

1.1.9. Arbitrageur

Arbitrage can be defined as a trading strategy that takes advantage of two or more securities being mispriced relative to each other. An arbitrageur takes no risk in making profit.

1.1.10. Speculator

This is an individual that takes a position in the market. Usually the individual is betting that the price of an asset will go up or that the price of an asset will go down.

1.1.11. Payoff

This is defined as the cash realized by the holder of an option or other derivative at the end of its life.

1.2. Ito's Formula

We considered a function f depending on the asset price S_t and time t. If the asset price S were a deterministic variable, we would simply expand $f(S_0+\Delta s, \Delta t)$ at $f(S, 0)$ in Taylor series:

$$\Delta f=\left(\frac{\partial f}{\partial t}\Delta t+\frac{1}{2}\frac{\partial^2 f}{\partial t^2}\Delta t^2+\cdots\right)+\left(\frac{\partial f}{\partial s}\Delta S+\frac{1}{2}\frac{\partial^2 f}{\partial s^2}\Delta S^2+\cdots\right)+\cdots \quad (1.1)$$

The Ito's calculus is a stochastic process equivalent to Newtonian differentiation calculus. In the limit $\Delta t \to 0$, terms Δt of higher order than 1 as in the ordinary differential calculus are considered small and can be omitted.

In case of lognormal random walk we can write that

$$\Delta S=S_0\mu\Delta t+S_0\sigma\sqrt{\Delta t\, Z} \quad (1.2)$$

because ΔS depends on Δt in case of random process. Then consider

$$(\Delta s)^2=(S_0\mu\Delta t+\sigma\sqrt{\Delta t S_0 Z})^2 \quad (1.3)$$

since Z is standard normal, Z^2 is distributed with gamma distribution with mean 1.

Therefore

$$E\left[(\sigma\sqrt{\Delta t Z}-\sigma^2 S_0\Delta t)^2\right]$$
$$=\sigma^2 S_0\Delta t E[Z^2]+0\left(t^{3/2}\right)=\sigma^2 S_0\Delta t+0\left(t^{3/2}\right) \quad (1.4)$$

In the limit $\Delta t \to 0$

$$d{S_1}^2=\sigma^2 {S^2}_0 dt \quad (1.5)$$

Therefore if f is a function dependent on S_t, it is also a stochastic process f_t such that

$$df_t=\frac{\partial f}{\partial t}dt+\frac{\partial f}{\partial s}dS_t+\frac{1}{2}\sigma^2 {S_0}^2\frac{\partial^2 f}{\partial s^2}dt \quad (1.6)$$

Or writing out ds_t, we obtain Ito's formula for the option, given the underlying asset is a stochastic process with

$$df_t=\mu S_0 dt+\sigma S_0 dw$$

$$df_t=\frac{\partial f}{\partial s}\sigma dw_t+\left(\frac{\partial f}{\partial t}+\mu S_0\frac{\partial f}{\partial s}+\frac{1}{2}\sigma^2 {S^2}_0\frac{\partial^2 f}{\partial s^2}\right)dt \quad (1.7)$$

The stochastic variable dw_t present in the formula means that the option price $f(S_t, t)$ also moves randomly.

2. The Binomial Model

This is defined as an iterative solution that models the price evolution over the whole option validity period. For some vanilla options such as American option, iterative model is the only choice since there is no known closed form solution that predicts its price over a period of time.

Black-Scholes model seems dominated the option pricing, but it is not the only popular model, the Cox-Ross-Rubinstein "Binomial" model has also a large popularity. The Cox-Ross-Rubinstein "Binomial" model has the Black-Scholes analytic formula as the limiting case as the number of steps tends to infinity.

Cox-Ross-Rubinstein [9] presented the binomial tree model in paper, "Option Pricing: A simplified approach" in 1979. The model is relatively simple and easy to understand, but it is an extremely powerful tool for pricing a wide range of options. They found a better stock movement model other than the geometric Brownian motion model applied by Black and Scholes, the binomial tree model. The tree specifies precisely all the possible future stock prices and the associated possibilities to obtain those prices.

The rate of return on the stock over each period can have two possible values: u with possibility q, or d with probability $(1-q)$. Thus, if the current stock price is S, the stock price at the end of the period will be either Su or Sd. The binomial model of the stock price movements is a discrete time model as opposed to the geometric Brownian motion model, which is a continuous time model.

2.1. Vanilla options

This is defined as a financial instrument that gives the holder of that option the right but not the obligation to buy or sell an underlying asset at a predetermined price within a given time frame. In other words, vanilla option is a normal call or put option that has standardized terms and no special or unusual features. It is generally traded on an exchange such as the Chicago Board Options Exchange. Examples of vanilla options are the American and European option.

2.1.1. American Option

This is defined as an option which can be exercised at any time up to and including the expiry date of the option. This added flexibility over European options results in American options having a value of at least equal to that of an identical European option, although in many cases the values are very similar as the optimal exercise date is often the expiry date.

2.1.2. European Option

This is defined as an option which is only exercisable at the expiry date of the option and can be valued using Black-Scholes option pricing formula. There are only five inputs to the classic Black-Scholes model: Spot price, Strike price, time until expiry, Interest rate and volatility. As such European options are typically the simple option to value. The dividend or yield of an underlying asset can also be an input to model. The term European option is confined to describing the exercise feature (i.e exercisable only on the expiry date) of the option and does not describe the geographic region of the underlying asset.

2.2. Binomial Model, an Alternative for Deriving Black-Scholes Model

In this section, we will show that given the risk neutral binomial process, we can derive the Black-Scholes equation from the risk-neutral expectation formula given below:

$$V_0 = e^{-rT}E_q[f] \tag{2.1}$$

Consider the single period binomial model. Let the current underlying price of the asset be S. The risk neutral expectation is given below:

$$E_q[V(S,0)] = qV(Su,\Delta t) + (1-q)V[Sd,\Delta t] \tag{2.2}$$

Expand $V(Su,\Delta t)$ in Taylor series:

$$V(Su,\Delta t) = V(S+S(u-1),\Delta t)$$

$$= V(S,\Delta t) + V'(S,\Delta t)S(u-1) + \frac{1}{2}V''(S,\Delta t)S^2(u-1)^2 + 0(u^3).$$

Expand $V(Sd,\Delta t)$ in Taylor series:

$$V(Sd,\Delta t) = V(S+S(d-1),\Delta t)$$

$$= V(S,\Delta t) + V'(S,\Delta t)S(d-1) + \frac{1}{2}V''(S,\Delta t)S^2(d-1)^2 + 0(d^3).$$

Put the values of $V(Su,\Delta t)$ and $V(Sd,\Delta t)$ in equation (2.2) we have

$$E_q[V(S,0)]$$

$$= q\left[V(S,\Delta t) + V'(S,\Delta t)S(u-1) + \tfrac{1}{2}V''(S,\Delta t)S^2(u-1)^2 + 0(u^3)\right]$$

$$+(1-q)\left[V(S,\Delta t) + V'(S,\Delta t)S(d-1) + \tfrac{1}{2}V''(S,\Delta t)S^2(d-1)^2 + 0(d^3)\right],$$

$$= q\left[V(S,\Delta t) + SuV'(S,\Delta t) - SV'(S,\Delta t) + \tfrac{1}{2}u^2V''(S,\Delta t)S^2 + \tfrac{1}{2}V''(S,\Delta t)S^2 - uV''(S,\Delta t)S^2\right]$$

$$+(1-q)\left[V(S,\Delta t) + V'(S,\Delta t)dS - SV'(S,\Delta t) + \tfrac{1}{2}d^2V''(S,\Delta t)S^2 + \tfrac{1}{2}V''(S,\Delta t)S^2 - dV''(S,\Delta t)S^2\right] \tag{2.3}$$

$$= q\,V(S,\Delta t) + SuV'(S,\Delta t)q - SV'(S,\Delta t)q + \tfrac{1}{2}u^2V''(S,\Delta t)S^2 - uV''(S,\Delta t)S^2 + V(S,\Delta t) + V'(S,\Delta t)dS$$

$$- SV'(S,\Delta t) + \tfrac{1}{2}d^2V''(S,\Delta t)S^2 + \tfrac{1}{2}V''(S,\Delta t)S^2 - dV''(S,\Delta t)S^2 - q\,V(S,\Delta t)$$

$$-qdV'(S,\Delta t)S + qV'(S,\Delta t)S - \tfrac{1}{2}qd^2V''(S,\Delta t)S^2 - \tfrac{1}{2}qV''(S,\Delta t)S^2 + qdV''(S,\Delta t)S^2$$

$$= V(S,\Delta t) + V'(S,\Delta t)S[q(u-1) - (1-q)(d-1)] + \frac{1}{2}V''(S,\Delta t)S^2(u-1)^2 + 0(u^3)$$

We will use the equality

$$qu + (1-q)d = e^{r\Delta t},$$

And by risk neutral argument, this must be equal to

$$V(S,0)e^{r\Delta t} = V(S,0)(1+r\Delta t) + 0(\delta t^2)$$

$$= V(S,\Delta t) + V'(S,\Delta t)S[e^{r\Delta t} - 1]$$

$$+\frac{1}{2}V''(S,\Delta t)S^2(u-1)^2 + 0(u^3)$$

$$= V(S,\Delta t) + V''(S,\Delta t)Sr\Delta t$$

$$+\frac{1}{2}V''(S,\Delta t)S^2\sigma^2\Delta t + 0\left(\Delta t^{3/2}\right)$$

By the risk neutral argument, this must be equal to

$$V(S,0)e^{r\Delta t} = V(S,0)(1+r\Delta t) + 0(\delta t^2)$$

Rearranging

$$V(S,\Delta t) - V(S,0) + \tfrac{1}{2}S^2\sigma^2V''(S,\Delta t) + rSV'(S,\Delta t)\Delta t \tag{2.4}$$

When limit $\Delta t \to 0$, we will finally get the Black-Scholes partial differential equation

$$\frac{\partial V}{\partial f} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{2.5}$$

2.3. Binomial Asset Price Process

The binomial model starts out with an extremely simple two state market model shown in the diagram below:

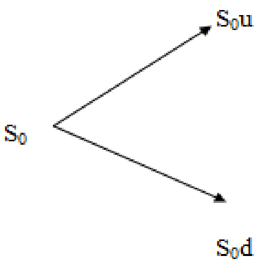

Figure 1. *One Period Binomial Model*

If S_0 is the spot price of a risky asset at time $t=0$ after some period of time T, it can only assume two distinct values. S_0u and S_0d, where u and d are real numbers such that $u>d$. Moreover we will assume the existence of a riskless asset with a constant yield r.

Thus, we can say that an investment of S_0 dollars at time $t=0$ yields S_0e^{rT} dollars at time $t=T$. The no arbitrage argument is also valid here, we must require that

$$S_0d < S_0e^{rT} < S_0u \text{ or } d < e^{rT} < u$$

If it is time, then it means that a riskless investment can be better, worse or even as well as a risky investment. If this is not time, then the risky asset is not risky at all. If $e^{rT} < d < u$, one would never prefer the risk-free asset to the risky asset; borrowing φ units of money at the risk-less rate r and buying the risky asset would yield a profit of at least $\varphi(d-e^{rT})>0$ at time $t=T$. If $e^{rT}=d$ such a market position (long asset, short bond) would yield a positive value. If $e^{rT}\geq u$ then market position will be reversed (long bond, short asset) yield a positive value.

Suppose now that the option yields f_u and f_d , the underlying asset go up and down respectively.

Consider a portfolio consisting of Δ units of the risky asset (e.g a stock) and φ units of risk-less asset (e.g a money market account) forms a replicating portfolio

$$\Delta S_0u + \varphi e^{rT} = f_u \tag{2.6}$$

$$\Delta S_0d + \varphi e^{-rT} = f_d \tag{2.7}$$

This is a system of two equations with two unknowns (Δ,φ) i.e there is a unique solution exist if and only if $u\neq d$

$$\Delta = \frac{f_u-f_d}{S_0u-S_0d} \tag{2.8}$$

$$\varphi = e^{-rT}\frac{uf_d-uf_u}{u-d} \tag{2.9}$$

Since the option payoff at $t=T$ is equal to that of this portfolio, the value of the portfolio must be equal to that of the option. Let's say the present value of the option is V_0

$$V_0 = \Delta S_0 + \varphi$$

Putting the values of Δ and φ, we will get

$$V_0 = \frac{f_u-f_d}{S_0u-S_0d}S_0 + e^{-rT}\frac{uf_d-uf_u}{u-d}$$

$$= \frac{f_u-f_d}{S_0(u-d)}S_0 + e^{-rT}\frac{uf_d-df_u}{u-d}$$

$$= \frac{f_u-f_d}{u-d} + e^{-rT}\frac{uf_d-df_u}{u-d} \text{ b}$$

Thus,

$$V_0 = \frac{f_u-f_d+e^{-rT}(uf_d-df_u)}{u-d} \tag{2.10}$$

We introduce a new variable

$$q = \frac{e^{rT}-d}{u-d}$$

The value of the option at $t=0$ can be expressed as

$$V_0 = e^{-rT}[qf_u + (1-q)f_d] \tag{2.11}$$

The no arbitrage argument guarantees that $0<q<1$. Thus the value of the option reduces to a certain kind of expectation formula

$$V_0 = e^{-rT}E_q[f],$$

where the expectation is taken under the probability measure given by q. This measure has the special property that if V_T is the value of option at $t=T$,

$$E_q[V_T] = (\Delta S_0 + \varphi)e^{rT},$$

$$E_q[V_T] = V_0e^{rT}$$

This probability measure is called risk-neutral probability measure.

2.3.1. Multi-Period Binomial Model

Multi-period binomial models applied to the same total period of time $T=N\Delta t$ as the number of periods increased, time step $\Delta t\to 0$ the distribution of $(\log S_T - \log S_0)$ approaches to normal distribution. In multi-period model the expiry of the option T is divided into two equal time-steps $T=2\Delta t$, a risky asset moves upward by a factor of u and downward by a factor of d.

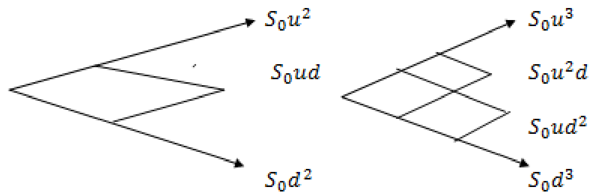

Figure 2. *Two and Three Period Binomial Models*

Thus recombining binomial tree has the end asset values (S_0u^2, S_0ud, S_0d^2) at time $t=T=2\Delta t$. Suppose now the option payoff function is $f(S,t)$. The three options at $t=T$ are given by

$$f_u = f(S_0u^2)$$

$$f_m = f(S_0ud)$$

$$f_d = f(S_0d^2)$$

Assuming the no-arbitrage and risk-neutral, we can apply the following formula

$$V_0 = e^{-rT}[qf_u + (1-q)f_d]$$

to each of the individual branches in this tree to obtain a value for the option step by step. At time $T = 2\Delta t$, we will get either of the two values of the option

$$V_1 = e^{-rT}[qf_u + (1-q)f_m] \quad (2.12)$$

$$V_1 = e^{-rT}[qf_u + (1-q)f_d] \quad (2.13)$$

Applying the formula once again

$$V_0 = e^{-r\Delta T}\left[qV_1{}'' + (1-q)V_1{}^d\right] \quad (2.14)$$

Inserting the values of $V_1{}^u, V_1{}^d$ we can write this as

$$V_0 = e^{-rT}[q^2 f(S_0u^2) + q(1-q)(S_0ud)f(S_0d^2)],$$

$$V_0 = e^{-rT}\sum_{j=0}^{2} q^j(1-q)^{\,2-j} f(S_0u^jd^{2-j}) \quad (2.15)$$

For the N-period model, where$T = N\Delta t$, we obtain

$$V_0 = e^{-rT}\sum_{j=0}^{N}\binom{N}{j} q^j(1-q)^{\,2-j} f(S_0u^jd^{2-j}) \quad (2.16)$$

The payoffs at each node in the N-period model can be expressed as function of the payoffs in $N+1$ period model:

$$f(S_0u^jd^{n-j})$$

$$= e^{-r\Delta t}\left(qf(S_0u^{j+1}d^{N-j}) + (1-q)f(S_0u^jd^{N+1-j})\right) \quad (2.17)$$

Substituting equation (2.17) into equation (2.16)

$$V_0 = e^{-r(T+\Delta t)}q^{N+1}f(S_0u^N)$$

$$+e^{-r(T+\Delta t)}\sum_{j=1}^{N}\left[\binom{N}{j} + \binom{N}{j-1}\right]\left(q^j(1-q)^{\,N-j}f(S_0u^jd^{N-j})\right)$$

$$+e^{-r(T+\Delta t)}(1-q)^{\,N+1}f(S_0d^N)$$

As we know

$$\binom{N}{j} + \binom{N}{j-1} = \binom{N+1}{j}.$$

The above equation then becomes:

$$V_0 = e^{(-r(N+1)\Delta t)}\sum_{j=0}^{N+1}\binom{N+1}{j} q^j(1-q)^{\,N+1-j} f\left(S_0u^jd^{N+1-j}\right) \quad (2.18)$$

which confirms that formula is also valid for $N+1$-period model. By the principle of induction, a formula in equation (2.18) is true for N-period nodes.

2.3.2. Approximating Continuous Time Prices with Discrete Time Models

The N-period model in the previous section is for discrete model, while the Black-Scholes derived by stochastic differential equation was continuous, what would happen if $N \to \infty$ or $\Delta t \to 0$?

Consider the value of the underlying asset (the stock) after n periods has passed. There has been a random number say X_n, 'up-jumps' and $n - X_n$, 'down jumps' the value of asset is then

$$S_n = S_0u^{X_n}d^{(n-X_n)} \quad (2.19)$$

The log normal process introduced involves a stock price $S_t = S_0e^{X_t}$, where X_t is given by its differential equation, it is a stochastic process with drift μt and variance$\sigma^2 t$.

Then the value of u, d will be the following

$$u = e^{\mu\Delta t + \sigma\sqrt{\Delta t}}$$

$$d = e^{\mu\Delta t - \sigma\sqrt{\Delta t}}$$

It is clear that $d < u$, if we assume that arbitrage requirement $d < e^{rT} < u$ satisfies then the asset price after n periods will be

$$S_n = S_0e^{(\mu\Delta t + \sigma\sqrt{\Delta t})(N - x_n)} \quad (2.20)$$

$$S_n = S_0e^{\mu t + (2x_n - n)\sigma\sqrt{\Delta t}} \quad (2.21)$$

If the limit $n \to \infty$, then $\mu t + (2x_n - n)\sigma\sqrt{\Delta t} \to x_n$? If it does, the binomial model can be used for approximating the lognormal process.

By using the values of u, d we can write q as

$$q = \frac{e^{r\Delta t} - d}{u - d}$$

$$q = \frac{e^{r\Delta t} - e^{\mu t - \sigma\sqrt{\Delta t}}}{e^{\mu t + \sigma\sqrt{\Delta t}} - e^{\mu t + \sigma\sqrt{\Delta t}}} \quad (2.22)$$

Expanding the denominator and numerator in Taylor series

$$q = \frac{r\Delta t - \left(\mu\Delta t - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t\right) + 0\left(\Delta t^{3/2}\right)}{\left(\mu\Delta t + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t\right) - \left(\mu\Delta t - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t\right) + 0\left(\Delta t^{3/2}\right)}$$

$$q = \frac{\sigma\sqrt{\Delta t} + \left(r - \mu - \frac{1}{2}\sigma^2\right)\Delta t + 0\left(\Delta t^{3/2}\right)}{2\sigma\sqrt{\Delta t} + 0\left(\Delta t^{3/2}\right)} \quad (2.23)$$

$$q = \frac{1}{2} + \frac{r - \mu - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t} + 0(\Delta t)$$

The random component of $S_n$is

$$R(S_n) = (2x_n - n)\sigma\sqrt{\Delta t} = (2x_n - n)\sigma\sqrt{\frac{t}{n}} = \frac{(2x_n - n)}{\sqrt{n}}\sigma\sqrt{t}$$

Let

$$Y_n = \frac{(2x_n - n)}{\sqrt{n}}$$

Now we will find the mean and variance of Y_n.

X_n can be described as sum of n independent Bernoulli random variables i.e random variables denoting the number of heads in flips of a coin, here the coin is a risk neutral coin with probability of heads equal to q.

The mean of x_n is nq and the variance of x_n is $nq(1-q$.

The mean of Y_n is then

$$E[Y_n] = \frac{(2E[x_n] - n)}{\sqrt{n}} = \frac{2nq - n}{\sqrt{n}} = \frac{n(2q-1)}{\sqrt{n}} = (2q-1)\sqrt{n}$$

Put the value of q in the above equation and we will get the mean of Y_n:

$$E[Y_n] = \frac{\sqrt{t}\left(r - \mu - \frac{1}{2}\sigma^2\right)}{\sigma} + 0(\Delta t)$$

The variance of Y_n is

$$Var[Y_n] = Var\left[\frac{2x_n}{\sqrt{n}}\right] = \frac{4var x_n}{n} = \frac{4nq(1-q)}{n} = 4q(1-q)$$

Put the value of q in the above equation and we will get the variance of Y_n

$$Var[Y_n] = 1 + 0(\Delta t) \tag{2.24}$$

In the limit $N \to \infty, \Delta t \to 0$ then $Y_n \sigma \sqrt{t}$ tends to a stochastic process Y_t distributed normally with mean $\left(r - \mu - \frac{1}{2}\sigma^2\right)t$ and the variance $\sigma^2 t$. We can write this as

$$Y_t = \left(r - \mu - \frac{1}{2}\sigma^2\right)t + \sigma w_t$$

where w_t is a Brownian motion we have shown that the discrete random process $S_n = S_0 e^{\mu t + (2x_n - n)\sigma\sqrt{\Delta t}}$ can be converted to continuous stochastic process $S_n = S_0 e^{\mu t + Y_t}$

Let

$$X_t = \mu t + Y_t, \text{ then we obtain}$$

$$X_t = \left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t \tag{2.25}$$

We can thus approximate lognormal asset price process with binomial model by setting

$$u = e^{\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}} \tag{2.26a}$$

$$d = e^{\left(r - \frac{1}{2}\sigma^2\right)\Delta t - \sigma\sqrt{\Delta t}} \tag{2.26b}$$

$$q = \frac{e^{r\Delta t} - d}{u - d} \tag{2.26c}$$

Let us now find dX_t, apply Ito's calculus to the above equation (2.25)

$$dX_t = \left(r - \frac{1}{2}\sigma^2\right)dt + \frac{1}{2}\sigma^2 dt + \sigma dw_t$$

$$dX_t = rdt - \frac{1}{2}\sigma^2 dt + \frac{1}{2}\sigma^2 dt + \sigma dw_t$$

$$dX_t = rdt + \sigma dw_t$$

This implies that in the risk neutral world the stochastic differential equation is

$$dX_t = \mu dt + \sigma dw_t \text{ as drift } \mu = r.$$

In other words

$$E[dx_t] = E[\mu dt + \sigma dw_t]$$

$$E\left[\frac{ds_t}{S_0}\right] = rdt \text{ or } E[S_t] = S_0 E[e^{X_t}]$$

$$E[S_t] = S_0 e^{rt}$$

The price of the stock is expected to grow at the risk-neutral rate r.

2.3.3. The Binomial Parameters

Here we will show the parameters of binomial model for continuous time prices using the lognormal price process.

Consider the binomial parameters which are defined in equation $u = e^{\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}}$, $d = e^{\left(r - \frac{1}{2}\sigma^2\right)\Delta t - \sigma\sqrt{\Delta t}}$ and $q = \frac{e^{r\Delta t} - d}{u - d}$ which are not only possible ways to construct a risk neutral binomial tree. The lognormal model is fully specified by the mean and variance of the random variable, $S_T = S_0 e^{X_T}$ or $e^{X_T} = \frac{S_T}{S_0}$, where X_t is given by $X_t = \left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t$. The variance of e^{X_T} is:

$$\begin{aligned} var[e^{X_T}] &= E[(e^{X_T})^2] - E[e^{X_T}]^2 \\ &= E[e^{2X_T}] - E[e^{X_T}]^2 \end{aligned} \tag{2.27}$$

where e^{X_T} has mean e^{rT} as shown in equation (2.27). To find the mean e^{2X_T}, we apply Ito's calculus:

$$X_t = \left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t, \text{ then}$$

$$2X_t = 2\left(r - \frac{1}{2}\sigma^2\right)t + 2\sigma W_t$$

To find dX_t

$$\begin{aligned} dX_t &= 2\left(r - \frac{1}{2}\sigma^2\right)dt + \frac{1}{2}(2\sigma)^2 dt + 2\sigma dW_t, \\ dX_t &= 2rdt - \sigma^2 dt + \frac{1}{2}(4\sigma^2)\,dt + 2\sigma dW_t, \\ dX_t &= 2rdt - \sigma^2 dt + 2\sigma^2 dt + 2\sigma dW_t, \\ dX_t &= (2r + \sigma^2)dt + 2\sigma dW_t. \end{aligned} \tag{2.28}$$

It means e^{2X_T} has mean $e^{(2r+\sigma^2)T}$ and the variance of e^{X_T} is given below:

$$var[e^{X_T}] = e^{2rT + \sigma^2 T} - E[e^{X_T}]^2 \tag{2.29}$$

Since $e^{X_T} = \frac{S_T}{S_0}$, thus we can write the mean and the variance as

$$E\left[\frac{S_T}{S_0}\right] = e^{rT} \tag{2.30}$$

$$var\left[\frac{S_T}{S_0}\right] = e^{(2r+\sigma^2)T} - E\left[\frac{S_T}{S_0}\right]^2 \tag{2.31}$$

We will apply mean and variance to one period binomial model with $T = \Delta t$ and limit $\Delta t \to 0$. In this model, the mean and variance are given below:

$$E\left[\frac{S_T}{S_0}\right] = qu + (1-q)d \tag{2.32}$$

$$var\left[\frac{S_T}{S_0}\right] = qu^2 + (1-q)d^2 - E\left[\frac{S_T}{S_0}\right]^2 \tag{2.33}$$

Comparing equations (2.30) and (2.32) we have that

$$e^{rT} = qu + (1-q)d,$$

where $T = \Delta t$, then $qu + (1-q)d = e^{r\Delta t}$.

Again comparing equations (2.31) and (2.33) we have that

$$qu^2 + (1-q)d^2 - E\left[\frac{S_T}{S_0}\right]^2 = e^{(2r+\sigma^2)T} - E\left[\frac{S_T}{S_0}\right]^2, \tag{2.34}$$

$$qu^2 + (1-q)d^2 = e^{(2r+\sigma^2)T}$$

Here again $T = \Delta t$

$$qu^2 + (1-q)d^2 = e^{(2r+\sigma^2)\Delta T} \tag{2.35}$$

But

$$e^{(2r+\sigma^2)\Delta T} = e^{2r\Delta t+\sigma^2\Delta t} \tag{2.36}$$

Therefore we have the following:

$$qu + (1-q)d = e^{r\Delta T} \tag{2.37}$$

$$qu^2 + (1-q)d^2 = e^{2r\Delta t+\sigma^2\Delta t} \tag{2.38}$$

Now we have two equations with three unknown variables, one variable can be chosen.

For example, $q = \frac{1}{2}$

Putting the value of $q = \frac{1}{2}$ in equation (2.37) yields

$$\frac{1}{2}u + \left(1 - \frac{1}{2}\right)d = e^{r\Delta t} \tag{2.39}$$

$$\frac{1}{2}u + \frac{1}{2}d = e^{r\Delta t} \tag{2.40}$$

$$\frac{1}{2}(u+d) = e^{r\Delta t} \tag{2.41}$$

Equation (2.41) becomes;

$$(u+d) = 2e^{r\Delta t} \tag{2.42}$$

Again substituting the value of $q = \frac{1}{2}$ into equation (2.38)

$$\frac{1}{2}u^2 + \left(1 - \frac{1}{2}\right)d^2 = e^{2r\Delta t+\sigma^2\Delta t} \tag{2.43}$$

$$\frac{1}{2}u^2 + \frac{1}{2}d^2 = e^{2r\Delta t+\sigma^2\Delta t} \tag{2.44}$$

$$\frac{1}{2}(u^2 + d^2) = e^{2r\Delta t+\sigma^2\Delta t} \tag{2.45}$$

Hence (2.45) yields;

$$(u^2 + d^2) = 2e^{2r\Delta t+\sigma^2\Delta t} \tag{2.46}$$

Now again we have two equations

$$(u+d) = 2e^{r\Delta t} \tag{2.47}$$

$$(u^2 + d^2) = 2e^{2r\Delta t+\sigma^2\Delta t} \tag{2.48}$$

From (2.47) and (2.48) we can get the value of u, d which are given below:

$$u = e^{r\Delta t}\left(1 + \sqrt{e^{\sigma^2\Delta t} - 1}\right) \tag{2.49}$$

$$d = e^{r\Delta t}\left(1 - \sqrt{e^{\sigma^2\Delta t} - 1}\right) \tag{2.50}$$

This proves that the binomial model approximates the lognormal price process. In the sequel we consider the convergence of the binomial option pricing model to the analytic option pricing model called "Black-Scholes Model" as follows:

2.4. Convergence of the Binomial Model to the Black-Scholes Model

The Black-Scholes formula for pricing European call option is given by

$$C_{T|0} = S_0\Phi(d_1) - K_{T|0}e^{-rT}\Phi(d_2) \tag{2.51}$$

where $\Phi(d)$ denotes the value of the cumulative Normal distribution function i.e the probability that $Z \leq d$ when $Z \sim N(0,1)$ is a standard normal variable and where

$$d_1 = \frac{\ln(S_0|K_{T|0}) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{2.52}$$

$$d_2 = \frac{\ln(S_0|K_{T|0}) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{2.53}$$

We can show that as the number n of the subintervals of the finite period $[0, T]$ increases indefinitely, the binomial formula for the value $C_{T|0}$ of the call option converges on Black-Scholes formula. We may begin by simplifying the binomial formula. Observe that for some outcomes there is

$$max\left(S_0u^jd^{n-j} - K_{T|0}\right) = 0 \tag{2.54}$$

Let a be the smallest number of upward movements of the underlying stock price that will ensure that the call option has a positive value, which is to say that it finishes in-the-money. Then $S_0 u^a d^{n-a} = K_{T|0}$; and only the binomial paths from $j = a$ onwards needs to be taken into account. Therefore the equation for the generalization n-sub periods i.e.

$$C_{T|0} = e^{-rT}\left\{\sum_{j=0}^{n}\frac{n!}{(n-j)!j!}p^j(1-p)^{n-j}C^{uj,d(n-j)}\right\} \quad (2.55)$$

$$= e^{-rT}E\left(C_{T|T}\right)$$

Equation (2.55) can be written as;

$$C_{T|0}$$

$$= e^{-rT}\left\{\sum_{j=a}^{n}\frac{n!}{(n-j)!j!}p^j(1-p)^{n-j}\left[S_0 u^j d^{n-j} - \right.\right. \quad (2.56)$$

$$K T 0$$

$$C_{T|0} = \left\{S_0 e^{-rT}\sum_{j=a}^{n}\frac{n!}{(n-j)!j!}p^j(1 - \right.$$

$$pn-jujdn-j \quad (2.57)$$

$$-K_{T|0}e^{-rT}\left\{\sum_{j=a}^{n}\frac{n!}{(n-j)!j!}p^j(1-p)^{n-j}\right\}$$

To demonstrate that this converges to equation (2.51) as $n \to \infty$, it must be shown that the terms in (2.57), (2.46) is associated with S_0 and $K_{T|0}e^{-TT}$, converge to $\Phi(d_1)$ and $\Phi(d_2)$ respectively.

The term associated with $K_{T|0}e^{-rT}$ is a simple binomial sum; and in the limit as $n \to \infty$, it converges to the partial integral of a standard normal distribution. The term associated with S_0 can be simplified so that it becomes a binomial sum that converges to a normal integral. Define the growth factor R by the equation $R^n = e^{-TT}$, then in reference to the equation that:

$$P = \frac{e^{TT}-d}{u-d} \; and \; 1-p = \frac{u-e^{TT}}{u-d} \quad (2.58)$$

It can be seen that within the context of the n-period binomial model, there is

$$P = \frac{R-d}{u-d} \; and \; 1-P = \frac{u-R}{u-d} \quad (2.59)$$

Now define: $P_* = \frac{u}{R^p}$ and $1 - P_* = \frac{d(1-p)}{R}$

Then the term associated with S_0 can be written as

$$\sum_{j=0}^{n}\frac{n!}{(n-j)!j!}{P_*}^j(1-P_*)^{n-j} \quad (2.60)$$

The task is now to replace the binomial sums as $n \to \infty$ by corresponding partial integrals of the standard normal distribution.

First observe that the condition $S_0 u^a d^{n-a} \backsimeq K_{T|0}$ can be solved to give

$$a = \frac{\ln(K_{T|0}|S_0) - n\ln d}{\ln(u/d)} + 0\left(n^{-1/2}\right) \quad (2.61)$$

Next let $S_T = S_0 u^j d^{n-j}$ be the stock price on expiry. This gives

$$\ln(S_T|S_0) = j\ln(u/d) + n\ln d \quad (2.62)$$

From which

$$E\{\ln(S_T|S_0)\} = E(j)\ln(u/d) + n\ln d \quad (2.63a)$$

and

$$V\{\ln(S_T|S_0)\} = V(j)\{\ln(u/d)\}^2 \quad (2.63b)$$

The equations (2.63a) and (2.63b) are solved to give respectively

$$E(j) = \frac{\{\ln(S_T|S_0)\} - n\ln d}{\ln(u/d)} \quad (2.64a)$$

and

$$V(j) = \frac{V\{\ln(S_T|S_0)\}}{\{\ln(u/d)\}^2} \quad (2.64b)$$

Now, the value a, which marks the first term in each of the binomial sums must be converted to a value that will serve as the limit of the corresponding integrals of the standard normal distribution.

The standardized value in question is $d = -\{a - E(j)\}/\sqrt{V(j)}$ to which a negative sign has been applied to ensure that the integral is over the interval $(-\infty, d]$ which accords with the usual tabulation of the cumulative normal distribution instead of the interval to$(-d, \infty]$, which would correspond more directly to the binomial summation from a to n.

Substituting (2.61), (2.64a) and (2.64b) into the expression for d gives

$$d = \frac{-\{a-E(j)\}}{\sqrt{V(j)}} = \frac{\ln(S_0|K_{T|0}) + E\{\ln(S_T|S_0)\}}{\sqrt{V\{\ln(S_T|S_0)\}}} - 0\left(n^{-1/2}\right) \quad (2.65)$$

As $n \to \infty$, the term of order $n^{-1/2}$ vanishes. Also the trajectory of the stock price converges to a geometric Brownian and from the note on continuous stochastic processes, we can gather the result that $V\{\ln(S_T|S_0)\} = \sigma^2 T$. This is regardless of the size of the drift parameter μ, which will vary with the values p and p_*. Therefore in the limit there is

$$d = \frac{\ln(S_0|K_{T|0}) + E\{\ln(S_T|S_0)\}}{\sigma\sqrt{T}} \quad (2.66)$$

It remains to show that

$$E\{\ln(S_T|S_0)\} = \begin{cases}(r-\sigma^2/2)T \; if \; the \; probability \; of \; u \; is \; p, \\ (r+\sigma^2/2)T \; if \; the \; probability \; of \; u \; is \; p_*\end{cases} \quad (2.67)$$

First, we consider $S_T|S_0 = \prod_{i=1}^{n}(S_i/S_{i-1})$, where S_n is synonymous with S_T. Since this is a product of sequence of independent and identically distributed random variables, there is

$$E(S_T|S_0) = \prod_{i=1}^{n}(S_i/S_{i-1}) = \{E(S_i/S_{i-1})\}^n \quad (2.68)$$

Moreover, since $S_i/S_{i-1} = u$ with probability p and $S_i/S_{i-1} = D$ with probability $(1-p)$, the expected value of this ratio is

$$\left.\begin{aligned} E(S_i/S_{i-1}) &= pu + (1-p)D \\ &= R \end{aligned}\right\} \quad (2.69)$$

where the second equality follows in view of the definitions of 2.58. Putting this back into (2.68) gives

$$E(S_T|S_0) = R^n \; and \; \ln\{E(S_T|S_0)\} = n \ln R \quad (2.70)$$

It follows from a property of the lognormal distribution that

$$\ln\{E(S_T|S_0)\} = E\left\{\{\ln(S_T|S_0)\} + \tfrac{1}{2}V\{\ln(S_T|S_0)\}\right\} \quad (2.71)$$

This is rearranged to give

$$\left.\begin{aligned} E\{\ln(S_T|S_0)\} &= \ln\{E(S_T|S_0)\} - \tfrac{1}{2}V\{\ln(S_T|S_0)\} \\ &= (r - \sigma^2/2)T \end{aligned}\right\} \quad (2.72)$$

The final equality follows on recalling the definitions that $R^n = e^{rT}$ and that $V\{\ln(S_T|S_0)\} = T\sigma^2$. This provides the first equality of 2.67

Now in pursuit of the second equality of (2.67) we must consider $(S_0|S_T) = \prod_{i=1}^{n}(S_{i-1}/S_i)$ which the inverse of the ratio in question is. In the manner of (2.68), there is

$$E(S_0|S_T) = \prod_{i=1}^{n}(S_{i-1}/S_i) = \{E(S_{i-1}/S_i)\}^n \quad (2.73)$$

However the expected value of the inverse ratio is

$$\left.\begin{aligned} E(S_{i-1}/S_i) &= p_* u^{-1} + (1-p_*)D^{-1} \\ & R^{-1} \end{aligned}\right\} \quad (2.74)$$

which follows in view of the definitions of P_* and $1 - p_*$of (2.59). Putting this back into (2.73) gives

$$E(S_0|S_T) = R^{-n} \; whence \; \ln\{E(S_0|S_T)\} = n \ln R \quad (2.75)$$

Now the object is to find $E\{\ln(S_T|S_0)\}$ from $\ln\{E(S_0|S_T)\}$. The property of the log normal distribution that gave 2.71 now gives

$$\left.\begin{aligned} \ln\{E(S_0|S_T)\} &= E\left\{\{\ln(S_0|S_T)\} + \tfrac{1}{2}V\{\ln(S_0|S_T)\}\right\} \\ &= -E\left\{\{\ln(S_T|S_0)\} + \tfrac{1}{2}V\{\ln(S_T|S_0)\}\right\} \end{aligned}\right\} \quad (2.76)$$

Here the second equality follows from the inversion of the ratio. This involves a change of sign of its logarithm, which affects the expected value on the RHS but not the variance. Rearranging the expression and using the result from 2.75 gives

$$\left.\begin{aligned} E\{\ln(S_T|S_0)\} &= n \ln R + \tfrac{1}{2}V\{\ln(S_T|S_0)\} \\ &= (r + \sigma^2/2) \end{aligned}\right\}. \quad (2.77)$$

This provides the second equality of (2.67).

3. Numerical Implementation and Examples

This section presents the implementation of binomial model for pricing vanilla options and examples as follows:

A1 The stock price S_{ti} at ti over time step δt can only take two possible values: either go up to $S_{ti}u$ or go down to $S_{ti}d$ at t_{i+1} with $0 < d < u$ where u is the factor of upward movement and d is the factor of downward movement.

A2 The probability of moving up between time t_i and t_{i+1} is p and therefore the probability of moving down is $(1-p)$.

A3 $$E\left(S_{t_{i+1}}|S_{t_i}\right) = S_{t_i}e^{r\delta t} \quad (3.1)$$

The probability p does not reflect the true probability of a stock moving up. It is an artificial probability reflecting the assumption A3. From assumptions A1 and A2, we have

$$E\left(S_{t_{i+1}}|S_{t_i}\right) = pS_{t_i}u + (1-p)S_{t_i}d \quad (3.2)$$

Equating this to $E\left(S_{t_{i+1}}|S_{t_i}\right)$ in assumption A3 we get

$$e^{r\delta t} = pu + (1-p)d \quad (3.3)$$

And solving for p we have that

$$p = \frac{(e^{r\delta t} - d)}{u-d} \quad (3.4)$$

In addition to this, at the expiry time $t = t_{M+1} = T$ there are $M + 1$ possible asset prices.

3.1. Numerical Examples

3.1.1. Example 1

The current security price is \$100. The exercise price on the option is \$110. It will either go up to \$150 or down to \$90. The riskless rate of interest is 5%. Maturity is 360 days, $T = 1$.

1. Calculate the price of the call option, the hedge ratio, and the probabilities of the up and down movements using Cox, Ross and Rubinstein model. Compare the result with the price calculated using BSM model. Calculate the present value of the ending payoff.
2. Calculate the weights for the replicating strategy, the ending payoff of the call option and the price of the call option. The bond price is \$100.

Solution

Table 1. *Parameters*

Security price	Exercise price	The payoff of the call option
150	110	$C_u = 40$
90	110	$C_d = 0$

The hedge ratio:

$$h = \frac{-(S_u - S_d)}{(C_u - C_d)} = \frac{-(150-90)}{(40-0)} = \frac{-60}{40} = -1.50$$

The ending payoff:

$$B = S_u + hC_u = 150 + (-1.50)40$$
$$= 150 - 60 = 90.0$$
$$B = S_d + hC_d = 90 + (-1.50)0$$
$$= 90 + 0 = 90$$
$$q = \frac{S_0(1+rT) - S_d}{S_u - S_d} = \frac{100(1+0.05(1)) - 90}{150-90} = 0.25$$
$$C = \frac{qC_u + (1-q)C_d}{(1+rT)} = \frac{0.25(40)+(1-0.25)0}{[1+(0.05)(1)]} = 9.5238 \cong 9.52$$

The option premium using Black-Scholes model:
The present value of the ending payoff:

$$B_0 = \frac{B}{(1+rT)} = \frac{90.0}{1.05\times 1} = \frac{90}{1.05} = 85.71$$
$$B_0 = S_0 + hC = 100 + (9.52)(-1.50) = 85.71$$

b. $W_a = \frac{C_u - C_d}{S_u - S_d} = \frac{-1}{h} = \frac{-1}{-1.50} = 0.6667 = 66.7\%$

$$W_b = \frac{C_d S_u - C_u S_d}{(S_u - S_d)P}; where\ P = Bond\ price = \$100$$
$$W_b = \frac{0(150) - 40(90)}{(150-90)100} = \frac{0-3600}{60\times 100} = \frac{-3600}{6000} = -0.60 = -60.0\%$$
$$C_u = W_a S_u + W_b P = \frac{66.7}{100}(150) + \left(-\frac{60.0}{100}\right)100 = 40$$
$$C_d = W_a S_d + W_b P = \frac{66.7}{100}(90) + \left(-\frac{60.0}{100}\right)100 = 0$$
$$C = W_a S_0 + W_b P_0 = \frac{66.7}{100}(100) + \left(-\frac{60.0}{100}\right)95 = 9.7$$

where the riskless rate of interest, $r = 5\%$ is being deducted from the bond price so that $P_0 = 95$

3.1.2. Example 2

Consider a standard option that expires in three months with an exercise price of $\$65$. Assume that the underlying stock pays no dividend, trades at $\$60$, and has a volatility of 30% per annum. The risk-free rate is 8% per annum.

We compute the values of both European and American style options using Binomial model against Black-Scholes model as we increase the number of steps with the following parameters:

$$S = 60, K = 65, T = 0.25, r = 0.08, \sigma = 0.30$$

The Black-Scholes price for call and put options are 2.1334 and 5.8463 respectively.

The results obtained are shown in Table 2 below

***Table 2.** The Comparative Results Analysis of the Binomial Model and Black Scholes Value ($B_C = 2.1334, B_P = 5.8463$) of the Standard Option*

N	European Call, E_C	American Call, A_C	European Put, E_p	American Put, A_p
20	2.1755	2.1755	5.8884	6.1531
40	2.1409	2.1409	5.8538	6.1283
60	2.1227	2.1227	5.8356	6.1178
80	2.1315	2.1315	5.8444	6.1246
100	2.1375	2.1375	5.8504	6.1280
120	2.1375	2.1375	5.8523	6.1287
140	2.1394	2.1394	5.8523	6.1282
160	2.1384	2.1384	5.8513	6.1274
180	2.1369	2.1369	5.8499	6.1262
200	2.1369	2.1369	5.8481	6.1249
220	2.1334	2.1334	5.8463	6.1237
240	2.1315	2.1315	5.8444	6.1225
260	2.1305	2.1305	5.8435	6.1224
280	2.1324	2.1324	5.8453	6.1235
300	2.1337	2.1337	5.8466	6.1243

3.1.3. Example 3

Consider pricing a vanilla option on a stock paying a known dividend yield with the following parameters:

$$S = 50, r = 0.1, T = 0.5, \sigma = 0.25, \gamma = \frac{1}{6}, \lambda = \frac{1}{20}$$

The results obtained are shown in Table 3 below

***Table 3.** Out of the Money, at the Money, and in the Money Vanilla Options on a Stock Paying a Known Dividend Yield.*

K	Ecall	Acall	Early exercise premium	Eput	Aput	Early exercise premium
30	18.97	20.50	1.53	0.004	0.004	0.00
45	6.06	6.47	0.41	1.37	1.49	0.12
50	3.32	3.42	0.10	3.38	3.78	0.40
55	1.62	1.63	0.01	6.40	7.31	0.91
70	0.11	0.11	0.00	19.19	21.35	2.16

3.1.4. Example 4

We value an European put option given two different current stock prices $S_0 = \$40$ and $S_0 = \$36$, two times to expiration, $T = 1$ and $T = 0.5$ and two volatility rates, $\sigma = 0.4$ and $\sigma = 0.2$. The strike price is $K = \$40$ and the risk-free rate of return is $r = 0.06$. The results obtained are shown in the Table 4 below.

***Table 4.** The Comparative Results Analysis of Finite Difference Method and Binomial Tree Pricing in the Context of Black-Scholes Model*

	Black-Scholes Price	Binomial Model	Binomial Model	Finite difference Implicit approach		Finite difference Explicit approach	
		European Price	American Price	European price	American price	European price	American price
$S_0 = 40$ $T = 1$; $\sigma = 0.4$	5.05962	5.05885	5.31792	5.04698	5.29048	5.10884	5.32085
$S_0 = 36$ $T = 1$; $\sigma = 0.4$	6.71140	6.71118	7.10897	6.70245	7.08102	6.75919	7.1119

	Black-Scholes Price	Binomial Model	Binomial Model	Finite difference Implicit approach		Finite difference Explicit approach	
		European Price	American Price	European price	American price	European price	American price
$S_0 = 40$ $T = 0.5;$ $\sigma = 0.4$	3.86569	3.86513	3.97775	3.85803	3.96196	3.90376	3.97910
$S_0 = 40$ $T = 1;$ $\sigma = 0.2$	2.06640	2.06600	2.31943	2.06137	2.30280	2.08613	2.32199

Binomial tree however is pretty much straight forward for implementation especially using Matlab.

3.2. Discussion of Results

We can see from Table 2 that Black-Scholes formula for the valuation of European call (Ecall) option can be used to value its counterpart American call (Acall) option for it is never optimal to exercise an American call option before expiration. As we increase the value of N, the value of the American put (Aput) option is higher than the corresponding European put (Eput) option as we can see from the above Table 2 because of the early exercise premium. Sometimes the early exercise of the American put option can be optimal. Table 2 also shows that the binomial model converges faster and close to the Black-Scholes value as the value of N is doubled. This method is very flexible in pricing vanilla options.

Table 3 shows that the American option on the dividend paying stock is always worth more its European counterpart. For a very deep in the money American option has a high early exercise premium. The premium of both put and Call option decreases as the option goes out of the money. The American and European call options are not worth the same as it is optimal to exercise the American call early on a dividend paying stock. For a deep out of the money American and European call options are worth the same; this is due to the fact that they might not be exercised early as they are worthless.

For Example 4, we assume 2000 time-steps for the binomial trees. For the implicit finite difference method, we have assumed 2000steps for the stock price and 50 time-steps that the American option can be exercised. For the explicit approach we assumed 200 steps for the stock price and 8000 time-steps due to the instable nature of this method. We have also used the control variant technique in the results obtained by the explicit approach, using the price of a similar European option obtained by the use of explicit finite difference approach as control variant, and Black-Scholes framework as an analytic solution. Table 4 shows that binomial model agrees with Black-Scholes values for a European price. Also binomial model and finite difference methods perform better when pricing American options. Hence we can deduce that binomial model converges faster to the Black-Scholes model when pricing a European option. This model is also good for pricing American option due to early exercise premium but path dependent options remain problematic.

4. Conclusion

Options come in many different flavours such as exotic options, fixed exercise time or early exercise options and so on. Binomial model is suited to dealing with some of these option flavors.

In general, binomial model has its advantages and disadvantages of use as listed below.

4.1. Advantages of Binomial Model

- Using the numerical approach of binomial model we can calculate the American option price as well s the European option price.
- Pharmaceutical companies benefit from the use of Binomial model method for real option valuation instead of older analysis as they deal with projects which have high risk and great uncertainty.
- Telecommunications sector which operates in a market that is highly volatile with projects in the industry that face considerable uncertainty and competition also uses Binomial model method to illustrate how the real option analysis can be used to help these industries make better investment decisions.
- The binomial model is much more capable of handling early exercise because it considers the cash flow at each time period rather than just the cash flows at expiration.
- Binomial model works for path dependent options.

4.2. Disadvantages of Binomial Model

- The binomial model is quite hard to adapt to more complex situations.
- Sometimes, though not at all times, the model fails to account for the value of managerial flexibility inherent in many types of project.
- The binomial model though can use a variant that allows the estimation of up and down movements in stock prices from the estimated variance; it can’t accurately determine what stock prices will be at the end of each period.
- Another major limitation of the binomial options pricing model is its slow speed.
- Also one of the key weaknesses is whether or not the assumptions made to simplify the model are likely to be true in a real life situation.
- They are not identical to the authentic items.

The binomial model is good for pricing options with early exercise opportunities, accurate, converges faster and it is relatively easy to implement but can be quite hard to adapt to more complex situations.

AMS Subject Classification: 34K50, 35A09, 91B02, 91B24, 91B25

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