**Applied and Computational Mathematics**
2015; 4(1-2): 10-14
Published online December 25, 2014 (http://www.sciencepublishinggroup.com/j/acm)
doi: 10.11648/j.acm.s.2015040102.13
ISSN: 2328-5605 (Print); ISSN: 2328-5613 (Online)

# A note on Zadeh's extension principle

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**To cite this article:**
Masamichi Kon. A Note on Zadeh's Extension Principle. *Applied and Computational Mathematics.* Special Issue: New Advances in Fuzzy Mathematics: Theory, Algorithms, and Applications. Vol. 4, No. 1-2, 2015, pp. 10-14. doi: 10.11648/j.acm.s.2015040102.13

**Abstract:** For a mapping, fuzzy sets obtained by Zadeh's extension principle are images of other fuzzy sets on the domain of the mapping under the mapping. Some relationships between images of level sets of one or two fuzzy sets under a mapping and another fuzzy set obtained from the one or two fuzzy sets by Zadeh's extension principle are known. In the present paper, the known results are extended to more general ones, and some useful results for applications are derived by the extended ones.

**Keywords:** Zadeh’s Extension Principle, Fuzzy Inner Product, Fuzzy Distance

## 1. Introduction

The concept of fuzzy sets has been primarily introduced for representing sets containing uncertainty or vagueness by Zadeh [9] as fuzzy set theory. Then, fuzzy set theory has been applied in various areas such as economics, management science, engineering, optimization theory, operations research, etc. [2, 3, 5–8]. Zadeh's extension principle [1, 5, 9] provides a natural way for extending the domain of a mapping. It is an important tool in the development of fuzzy arithmetic and other areas. Let $f : X \times Y \to Z$ be a mapping, and let $\tilde{a}$ and $\tilde{b}$ be fuzzy sets on $X$ and $Y$, respectively. In addition, let $f(\tilde{a}, \tilde{b})$ be the fuzzy set on $Z$ obtained from $\tilde{a}$ and $\tilde{b}$ by Zadeh's extension principle. In [4], relationships between $f([\tilde{a}]_\alpha, [\tilde{b}]_\alpha)$ and $[f(\tilde{a}, \tilde{b})]_\alpha$ are investigated, where $[\tilde{a}]_\alpha$, $[\tilde{b}]_\alpha$, and $[f(\tilde{a}, \tilde{b})]_\alpha$ are the $\alpha$-level sets of $\tilde{a}$, $\tilde{b}$, and $f(\tilde{a}, \tilde{b})$, respectively.

In the present paper, the results in [4] are extended to more general ones, and some useful results for applications are derived by the extended ones.

## 2. Preliminaries

In this section, some notations are presented.

Let $\mathbb{R}$ and $\mathbb{C}$ be sets of all real and complex numbers, respectively. For $a, b \in \mathbb{R}$, we set $[a, b] = \{x \in \mathbb{R} : a \le x \le b\}$, $[a, b[ = \{x \in \mathbb{R} : a \le x < b\}$, $]a, b] = \{x \in \mathbb{R} : a < x \le b\}$, and $]a, b[ = \{x \in \mathbb{R} : a < x < b\}$.

Throughout the present paper, let $X$, $Y$, and $Z$ be nonempty sets. We identify a fuzzy set $\tilde{a}$ on $X$ with its membership function $\tilde{a} : X \to [0,1]$. Let $\mathcal{F}(X)$ be the set of all fuzzy sets on $X$.

For $\tilde{a} \in \mathcal{F}(X)$ and $\alpha \in ]0,1]$, the set

$$[\tilde{a}]_\alpha = \{x \in X : \tilde{a}(x) \ge \alpha\}$$

is called the $\alpha$-level set of $\tilde{a}$.

For a crisp set $S \subset X$, the function $c_S : X \to \{0,1\}$ defined as

$$c_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S \end{cases}$$

for each $x \in X$ is called the indicator function of $S$.

A fuzzy set $\tilde{a} \in \mathcal{F}(X)$ can be represented as

$$\tilde{a} = \sup_{\alpha \in ]0,1]} \alpha c_{[\tilde{a}]_\alpha},$$

which is well-known as the decomposition theorem; see, for example, [1].

We set

$$\mathcal{P}(X) = \left\{\{S_\alpha\}_{\alpha \in ]0,1]} : S_\alpha \subset X, \alpha \in ]0,1]\right\},$$
$$\mathcal{S}(X) = \left\{\{S_\alpha\}_{\alpha \in ]0,1]} \in \mathcal{P}(X) : S_\beta \supset S_\gamma \text{ for } \beta, \gamma \in ]0,1] \text{ with } \beta < \gamma\right\},$$

and define $M_X : \mathcal{P}(X) \to \mathcal{F}(X)$ as

$$M_X\left(\{S_\alpha\}_{\alpha \in ]0,1]}\right) = \sup_{\alpha \in ]0,1]} \alpha c_{S_\alpha}$$

for each $\{S_\alpha\}_{\alpha \in ]0,1]} \in \mathcal{P}(X)$. For $\{S_\alpha\}_{\alpha \in ]0,1]} \in \mathcal{P}(X)$ and $x \in X$, it follows that

$$M_X(\{S_\alpha\}_{\alpha\in]0,1]})(x) = \sup_{\alpha\in]0,1]} \alpha c_{S_\alpha}(x) = \sup\{\alpha \in ]0,1] : x \in S_\alpha\},$$

where $\sup\emptyset = 0$. The decomposition theorem can be represented as

$$\tilde{a} = M_X(\{[\tilde{a}]_\alpha\}_{\alpha\in]0,1]})$$

for $\tilde{a} \in \mathcal{F}(X)$.

When $\tilde{a} = M_X(\{S_\alpha\}_{\alpha\in]0,1]})$ for $\tilde{a} \in \mathcal{F}(X)$ and $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{P}(X)$, $\tilde{a}$ is called the fuzzy set generated by $\{S_\alpha\}_{\alpha\in]0,1]}$, and $\{S_\alpha\}_{\alpha\in]0,1]}$ is called the generator of $\tilde{a}$.

# 3. Main Results

In this section, the results in [4] are extended to more general ones.

In [4], relationships between images of level sets of one or two fuzzy sets under a mapping and another fuzzy set obtained from the one or two fuzzy sets by Zadeh's extension principle are investigated. See [1, 4] for Zadeh's extension principle. The purpose of this section is to investigate relationships between images of generators of fuzzy sets under a mapping and another fuzzy set obtained from the fuzzy sets by Zadeh's extension principle.

We define images of fuzzy sets under a mapping by Zadeh's extension principle.

*Definition 1:*

(i) For $f : X \to Y$ and $\tilde{a} \in \mathcal{F}(X)$, $f(\tilde{a}) \in \mathcal{F}(Y)$ is defined as

$$f(\tilde{a})(y) = \sup_{x\in f^{-1}(y)} \tilde{a}(x)$$

for each $y \in Y$.

(ii) For $f : X \times Y \to Z$, $\tilde{a} \in \mathcal{F}(X)$, and $\tilde{b} \in \mathcal{F}(Y)$, $f(\tilde{a},\tilde{b}) \in \mathcal{F}(Z)$ is defined as

$$f(\tilde{a},\tilde{b})(z) = \sup_{(x,y)\in f^{-1}(z)} \tilde{a}(x)\wedge\tilde{b}(y)$$

for each $z \in Z$, where $\tilde{a}(x)\wedge\tilde{b}(y) = \min\{\tilde{a}(x),\tilde{b}(y)\}$.

*Proposition 1:[4]* Let $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{P}(X)$, and let $\tilde{a} = M_X(\{S_\alpha\}_{\alpha\in]0,1]}) \in \mathcal{F}(X)$. Then,

$$S_\alpha \subset [\tilde{a}]_\alpha$$

for any $\alpha \in ]0,1]$.

*Proposition 2:* Let $f : X \to Y$, and let $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{P}(X)$. In addition, let $\tilde{a} = M_X(\{S_\alpha\}_{\alpha\in]0,1]}) \in \mathcal{F}(X)$. Then,

$$f(\tilde{a}) = M_Y(\{f(S_\alpha)\}_{\alpha\in]0,1]}) = \sup_{\alpha\in]0,1]} \alpha c_{f(S_\alpha)}.$$

*Proof:* Let $y \in Y$. Then, we have

$$f(\tilde{a})(y) = \sup_{x\in f^{-1}(y)} \tilde{a}(x)$$

$$= \sup_{x\in f^{-1}(y)} \sup_{\alpha\in]0,1]} \alpha c_{S_\alpha}(x)$$

$$= \sup_{\alpha\in]0,1]} \sup_{x\in f^{-1}(y)} \alpha c_{S_\alpha}(x)$$

$$= \sup_{\alpha\in]0,1]} \alpha \sup_{x\in f^{-1}(y)} c_{S_\alpha}(x)$$

$$= \sup_{\alpha\in]0,1]} \alpha c_{f(S_\alpha)}(y).$$

*Remark 1:* Consider the same settings as in Proposition 2. It follows that

$$f(S_\alpha) \subset [f(\tilde{a})]_\alpha$$

for any$\alpha \in ]0,1]$ from Proposition 1. However,

$$f(S_\alpha) \neq [f(\tilde{a})]_\alpha$$

in general. Proposition 3.1 in [4] shows the case $\{S_\alpha\}_{\alpha\in]0,1]} = \{[\tilde{a}]_\alpha\}_{\alpha\in]0,1]}$. In this sense, Proposition 2 is an extension of Proposition 3.1 in [4].

*Proposition 3:* Let $f : X \times Y \to Z$, and let $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(X)$ and $\{T_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(Y)$. In addition, let $\tilde{a} = M_X(\{S_\alpha\}_{\alpha\in]0,1]}) \in \mathcal{F}(X)$ and $\tilde{b} = M_Y(\{T_\alpha\}_{\alpha\in]0,1]}) \in \mathcal{F}(Y)$. Then,

$$f(\tilde{a},\tilde{b}) = M_Z(\{f(S_\alpha,T_\alpha)\}_{\alpha\in]0,1]}) = \sup_{\alpha\in]0,1]} \alpha c_{f(S_\alpha,T_\alpha)}.$$

*Proof:* Let $z \in Z$. Then, it follows that

$$f(\tilde{a},\tilde{b})(z) = \sup_{(x,y)\in f^{-1}(z)} \tilde{a}(x)\wedge\tilde{b}(y)$$

$$= \sup_{(x,y)\in f^{-1}(z)} \left[\left\{\sup_{\alpha\in]0,1]} \alpha c_{S_\alpha}(x)\right\} \wedge \left\{\sup_{\alpha\in]0,1]} \alpha c_{T_\alpha}(y)\right\}\right]$$

and

$$\sup_{\alpha\in]0,1]} \alpha c_{f(S_\alpha,T_\alpha)}(z) = \sup_{\alpha\in]0,1]} \left[\sup_{(x,y)\in f^{-1}(z)} \alpha c_{S_\alpha}(x)\wedge\alpha c_{T_\alpha}(y)\right]$$

$$= \sup_{(x,y)\in f^{-1}(z)} \left[\sup_{\alpha\in]0,1]} \alpha c_{S_\alpha}(x)\wedge\alpha c_{T_\alpha}(y)\right].$$

Thus, it is sufficient to show that

$$\left\{\sup_{\alpha\in]0,1]} \alpha c_{S_\alpha}(x)\right\}\wedge\left\{\sup_{\alpha\in]0,1]} \alpha c_{T_\alpha}(y)\right\} = \sup_{\alpha\in]0,1]} \alpha c_{S_\alpha}(x)\wedge\alpha c_{T_\alpha}(y). \quad (1)$$

We set

$$\alpha_0 = \sup_{\alpha\in]0,1]} \alpha c_{S_\alpha}(x), \quad \beta_0 = \sup_{\alpha\in]0,1]} \alpha c_{T_\alpha}(y).$$

If $\alpha_0 = 0$, then $\alpha c_{S_\alpha}(x) = 0$ for any $\alpha \in ]0,1]$. If $\beta_0 = 0$, then $\alpha c_{T_\alpha}(y) = 0$ for any $\alpha \in ]0,1]$. Therefore, (1) holds if $\alpha_0\wedge\beta_0 = 0$. Suppose that $\alpha_0\wedge\beta_0 > 0$. From the definition of $\alpha_0$, it follows that $x \in S_\alpha$ for any $\alpha \in ]0,\alpha_0[$, and that $x \notin S_\alpha$ for any $\alpha \in ]\alpha_0, 1]$. From the definition of $\beta_0$, it follows that $y \in T_\alpha$ for any $\alpha \in ]0,\beta_0[$, and that $y \notin T_\alpha$ for any $\alpha \in ]\beta_0, 1]$. Therefore, since

$$\alpha c_{S_\alpha}(x)\wedge\alpha c_{T_\alpha}(y) = \begin{cases} \alpha & \text{if } \alpha \in ]0,\alpha_0\wedge\beta_0[, \\ \alpha_0\wedge\beta_0 \text{ or } 0 & \text{if } \alpha = \alpha_0\wedge\beta_0, \\ 0 & \text{if } \alpha \in ]\alpha_0\wedge\beta_0, 1], \end{cases}$$

we have

$$\sup_{\alpha\in]0,1]} \alpha c_{S_\alpha}(x) \wedge \alpha c_{T_\alpha}(y) = \alpha_0 \wedge \beta_0.$$

*Remark 2*: Consider the same settings as in Proposition 3. It follows that

$$f(S_\alpha, T_\alpha) \subset [f(\tilde{a}, \tilde{b})]_\alpha$$

for any $\alpha \in ]0,1]$ from Proposition 1. However,

$$f(S_\alpha, T_\alpha) \neq [f(\tilde{a}, \tilde{b})]_\alpha$$

in general. Proposition 3.2 in [4] shows the case $\{S_\alpha\}_{\alpha\in]0,1]} = \{[\tilde{a}]_\alpha\}_{\alpha\in]0,1]}$ and $\{T_\alpha\}_{\alpha\in]0,1]} = \{[\tilde{b}]_\alpha\}_{\alpha\in]0,1]}$. In this sense, Proposition 3 is an extension of Proposition 3.2 in [4]. Moreover, Proposition 3.3 in [4] shows that

$$f([\tilde{a}]_\alpha, T_\alpha) = [f(\tilde{a}, \tilde{b})]_\alpha$$

for any $\alpha \in ]0,1]$ if and only if $f^{-1}(z) = \emptyset$ or $\sup_{(x,y)\in f^{-1}(z)} \tilde{a}(x) \wedge \tilde{b}(y)$ is attained for any $z \in Z$.

The following proposition is obtained from Proposition 3.

*Proposition 4:* Let $m \geq 2$, and let $X_i, i = 1,2,\dots,2m-1$ be nonempty sets, and let $f_i : X_{2i-1} \times X_{2i} \to X_{2i+1}, i = 1,2,\dots,m-1$. In addition, let $\{S_{1\alpha}\}_{\alpha\in]0,1]} \in \mathcal{S}(X_1)$, and let $\tilde{a}_1 = M_{X_1}(\{S_{1\alpha}\}_{\alpha\in]0,1]}) \in \mathcal{F}(X_1)$, and let $\{S_{2i-2,\alpha}\}_{\alpha\in]0,1]} \in \mathcal{S}(X_{2i-2}), i = 2,3,\dots,m$, and let $\tilde{a}_i = M_{X_{2i-2}}\left(\{S_{2i-2,\alpha}\}_{\alpha\in]0,1]}\right) \in \mathcal{F}(X_{2i-2}), i = 2,3,\dots,m$. Then,

$$\begin{aligned} & f_{m-1}(\cdots f_3(f_2(f_1(\tilde{a}_1, \tilde{a}_2), \tilde{a}_3), \tilde{a}_4) \cdots, \tilde{a}_m) \\ &= M_{X_{2m-1}}(\{f_{m-1}(\cdots f_3(f_2(f_1(S_{1\alpha}, S_{2\alpha}), S_{4\alpha}), S_{6\alpha}) \\ &\qquad \cdots, S_{2m-2,\alpha})\}_{\alpha\in]0,1]}) \\ &= \sup_{\alpha\in]0,1]} \alpha c_{f_{m-1}(\cdots f_3(f_2(f_1(S_{1\alpha}, S_{2\alpha}), S_{4\alpha}), S_{6\alpha}) \cdots, S_{2m-2,\alpha})}. \end{aligned}$$

*Remark 3:* Consider the same settings as in Proposition 4. If $\{S_{1\alpha}\}_{\alpha\in]0,1]} = \{[\tilde{a}_1]_\alpha\}_{\alpha\in]0,1]}$ and $\{S_{2i-2,\alpha}\}_{\alpha\in]0,1]} = \{[\tilde{a}_i]_\alpha\}_{\alpha\in]0,1]}, i = 2,3,\dots,m$, then we have

$$\begin{aligned} & f_{m-1}(\cdots f_3(f_2(f_1(\tilde{a}_1, \tilde{a}_2), \tilde{a}_3), \tilde{a}_4) \cdots, \tilde{a}_m) \\ &= M_{X_{2m-1}}(\{f_{m-1}(\cdots f_3(f_2(f_1([\tilde{a}_1]_\alpha, [\tilde{a}_2]_\alpha), [\tilde{a}_3]_\alpha), [\tilde{a}_4]_\alpha) \\ &\qquad \cdots, [\tilde{a}_m]_\alpha)\}_{\alpha\in]0,1]}) \\ &= \sup_{\alpha\in]0,1]} \alpha c_{f_{m-1}(\cdots f_3(f_2(f_1([\tilde{a}_1]_\alpha, [\tilde{a}_2]_\alpha), [\tilde{a}_3]_\alpha), [\tilde{a}_4]_\alpha) \cdots, [\tilde{a}_m]_\alpha)}. \end{aligned}$$

However, it does not follow from Proposition 3.2 in [4] by the statements in Remark 2.

# 4. Applications

In this section, some useful results for applications are derived based on the results in the previous section.

We define a fuzzy inner product by Zadeh's extension principle.

*Definition 2:* Let $V$ be a complex inner product space equipped with an inner product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$. For $x, y \in V$, $\langle x, y\rangle \in \mathbb{C}$ is the inner product of $x$ and $y$. For $\tilde{a}, \tilde{b} \in \mathcal{F}(V)$, $\langle \tilde{a}, \tilde{b}\rangle \in \mathcal{F}(\mathbb{C})$ defined as

$$\langle \tilde{a}, \tilde{b}\rangle(x) = \sup_{\langle y,z\rangle = x} \tilde{a}(y) \wedge \tilde{b}(z)$$

for each $x \in \mathbb{C}$ is called the fuzzy inner product of $\tilde{a}$ and $\tilde{b}$.

Applying Proposition 3 to the fuzzy inner product, the following proposition is obtained.

*Proposition 5:* Let $V$ be a complex inner product space equipped with an inner product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$, and let $\{S_\alpha\}_{\alpha\in]0,1]}, \{T_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(V)$. In addition, let $\tilde{a} = M_V(\{S_\alpha\}_{\alpha\in]0,1]}), \tilde{b} = M_V(\{T_\alpha\}_{\alpha\in]0,1]}) \in \mathcal{F}(V)$. Then,

$$\langle \tilde{a}, \tilde{b}\rangle = M_{\mathbb{C}}(\{\langle S_\alpha, T_\alpha\rangle\}_{\alpha\in]0,1]}) = \sup_{\alpha\in]0,1]} \alpha c_{\langle S_\alpha, T_\alpha\rangle}.$$

We define a fuzzy distance by Zadeh's extension principle.

*Definition 3:* Let $X$ be a metric space equipped with a distance function $d : X \times X \to \mathbb{R}$. For $x, y \in X$, $d(x, y) \in \mathbb{R}$ is the distance between $x$ and $y$. For $\tilde{a}, \tilde{b} \in \mathcal{F}(X)$, $d(\tilde{a}, \tilde{b}) \in \mathcal{F}(\mathbb{R})$ defined as

$$d(\tilde{a}, \tilde{b})(x) = \sup_{d(y,z) = x} \tilde{a}(y) \wedge \tilde{b}(z)$$

for each $x \in \mathbb{R}$ is called the fuzzy distance between $\tilde{a}$ and $\tilde{b}$.

Applying Proposition 3 to the fuzzy distance, the following proposition is obtained.

*Proposition 6:* Let $X$ be a metric space equipped with a distance function $d : X \times X \to \mathbb{R}$. Let $\{S_\alpha\}_{\alpha\in]0,1]}, \{T_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(X)$, and let $\tilde{a} = M_X(\{S_\alpha\}_{\alpha\in]0,1]}), \tilde{b} = M_X(\{T_\alpha\}_{\alpha\in]0,1]}) \in \mathcal{F}(X)$. Then,

$$d(\tilde{a}, \tilde{b}) = M_{\mathbb{R}}(\{d(S_\alpha, T_\alpha)\}_{\alpha\in]0,1]}) = \sup_{\alpha\in]0,1]} \alpha c_{d(S_\alpha, T_\alpha)}.$$

We define a binary operation of fuzzy sets by Zadeh's extension principle.

*Definition 4:* Assume that $X$ is equipped with a binary operation $*$. For $x, y \in X$, $x * y \in X$. For any $\tilde{a}, \tilde{b} \in \mathcal{F}(X)$, we define $\tilde{a} * \tilde{b} \in \mathcal{F}(X)$ as

$$(\tilde{a} * \tilde{b})(x) = \sup_{y * z = x} \tilde{a}(y) \wedge \tilde{b}(z)$$

for each $x \in X$.

Applying Proposition 4 to the binary operation of fuzzy sets, the following proposition is obtained.

*Proposition 7:* Assume that $X$ is equipped with a binary operation $*$. Let $\{S_{i\alpha}\}_{\alpha\in]0,1]} \in \mathcal{S}(X), i = 1,2,\dots,m$, and let $\tilde{a}_i = M_X(\{S_{i\alpha}\}_{\alpha\in]0,1]}) \in \mathcal{F}(X), i = 1,2,\dots,m$. Then,

$$\begin{aligned} & (\cdots((\tilde{a}_1 * \tilde{a}_2) * \tilde{a}_3) \cdots * \tilde{a}_m) \\ &= M_X(\{(\cdots(((S_{1\alpha} * S_{2\alpha}) * S_{3\alpha}) * S_{4\alpha}) \\ &\qquad \cdots * S_{m-1,\alpha}) * S_{m\alpha}\}_{\alpha\in]0,1]}) \\ &= \sup_{\alpha\in]0,1]} \alpha c_{(\cdots(((S_{1\alpha} * S_{2\alpha}) * S_{3\alpha}) * S_{4\alpha}) \cdots * S_{m-1,\alpha}) * S_{m\alpha}}. \end{aligned}$$

In the rest of the present paper, let $X$ be a metric space equipped with a distance function $d: X \times X \to \mathbb{R}$, and discuss the application of the obtained results to a fuzzy location problem.

A problem to locate a single facility is called a single facility location problem. Let $a_i \in X, i = 1,2,\ldots,m$ be demand points. Demand points are fixed points which represent locations of customers of the facility to be located. For each $i \in \{1,2,\ldots,m\}$, let $w_i > 0$ be a weight associated with a demand point $a_i$. Let $x \in X$ be a variable location of the facility, and let $S \subset X, S \neq \emptyset$ be the feasible region which the facility can be located. Then, the crisp minisum location problem is formulated as

$$(\mathrm{P}) \qquad \left| \begin{array}{ll} \min & \sum_{i=1}^{m} w_i d(a_i, x) \\ \mathrm{s.t.} & x \in S. \end{array} \right.$$

Replacing demand points $a_i, i = 1,2,\ldots,m$ in (P) by fuzzy demand points $\tilde{a}_i \in \mathcal{F}(X), i = 1,2,\ldots,m$, the fuzzy minisum location problem is formulated. Fuzzy demand points are fuzzy sets which represent locations of customers of the facility to be located. Assume that $\mathcal{F}(\mathbb{R})$ is equipped with an ordering of some kind. Then, the fuzzy minisum location problem is formulated as

$$(\tilde{\mathrm{P}}) \qquad \left| \begin{array}{ll} \min & \sum_{i=1}^{m} w_i d(\tilde{a}_i, x) \\ \mathrm{s.t.} & x \in S, \end{array} \right.$$

where $d(\tilde{a}_i, x) = d(\tilde{a}_i, c_{\{x\}})$, and, for $\tilde{a} \in \mathcal{F}(\mathbb{R})$ and $\lambda \in \mathbb{R}$,

$$(\lambda \tilde{a})(y) = \sup_{\lambda z = y} \tilde{a}(z)$$

for each $y \in \mathbb{R}$ by Zadeh's extension principle. From the decomposition theorem and Propositions 2, 6, and 7, we have

$$\sum_{i=1}^{m} w_i d(\tilde{a}_i, x) = M_{\mathbb{R}}\left(\left\{\sum_{i=1}^{m} w_i d([\tilde{a}_i]_\alpha, x)\right\}_{\alpha \in ]0,1]}\right)$$

$$= \sup_{\alpha \in ]0,1]} \alpha c_{\sum_{i=1}^{m} w_i d([\tilde{a}_i]_\alpha, x)}.$$

It means that the objective function of $(\tilde{\mathrm{P}})$ can be primarily dealt with for general fuzzy sets, and the necessity and importance of the obtained results are revealed.

## 5. Examples

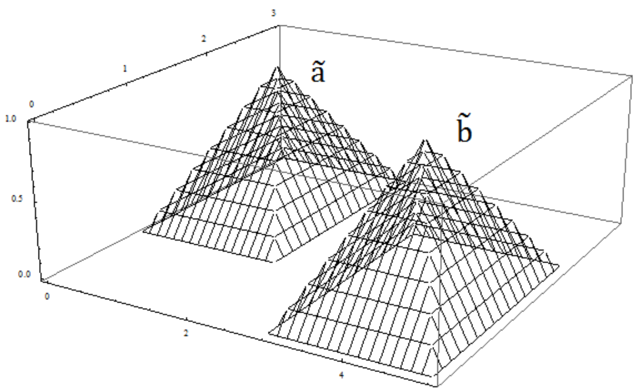


***Figure 1.*** $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R}^2)$.

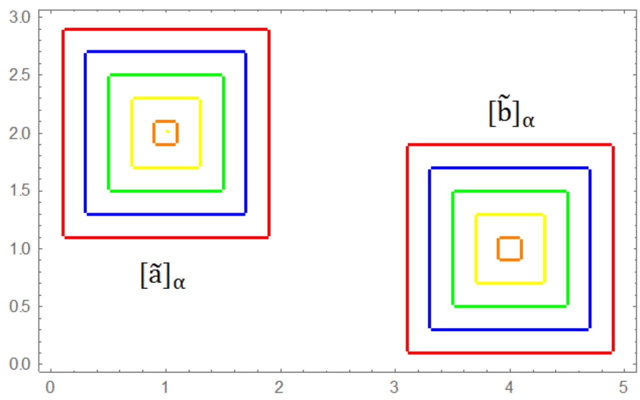


***Figure 2.*** $[\tilde{a}]_\alpha$ *and* $[\tilde{b}]_\alpha$.

In this section, examples of the fuzzy inner product and the fuzzy distance defined in the previous section are presented.

Let $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R}^2)$ be fuzzy sets defined as

$$\tilde{a}(y,z) = \min\{\max\{0, 1 - |y - 1|\}, \max\{0, 1 - |z - 2|\}\}$$

and

$$\tilde{b}(y,z) = \min\{\max\{0, 1 - |y - 4|\}, \max\{0, 1 - |z - 1|\}\}$$

for each $(y,z) \in \mathbb{R}^2$ (Fig. 1). Assume that $\mathbb{R}^2$ is equipped with the canonical inner product $\langle \cdot,\cdot \rangle: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ and the Euclidean distance $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$. For each $\alpha \in ]0,1]$, since $[\tilde{a}]_\alpha = [\alpha, 2-\alpha] \times [1+\alpha, 3-\alpha]$ and $[\tilde{b}]_\alpha = [3+\alpha, 5-\alpha] \times [\alpha, 2-\alpha]$ (Fig. 2), it follows that

$$\langle [\tilde{a}]_\alpha, [\tilde{b}]_\alpha \rangle = [2\alpha^2 + 4\alpha, 2\alpha^2 - 12\alpha + 16],$$

and that

$$d([\tilde{a}]_\alpha, [\tilde{b}]_\alpha) = [1 + 2\alpha, \sqrt{8\alpha^2 - 32\alpha + 34}]$$

for $\alpha \in ]0, 0.5]$ and

$$d([\tilde{a}]_\alpha, [\tilde{b}]_\alpha) = [\sqrt{8\alpha^2 + 2}, \sqrt{8\alpha^2 - 32\alpha + 34}]$$

for $\alpha \in ]0.5, 1]$. From the decomposition theorem and Proposition 5, we have

$$\langle \tilde{a}, \tilde{b} \rangle(x) = \begin{cases} \frac{1}{2}\sqrt{2x+4} - 1 & \text{if } x \in [0,6], \\ -\frac{1}{2}\sqrt{2x+4} + 3 & \text{if } x \in ]6,16], \\ 0 & \text{otherwise} \end{cases}$$

for each $x \in \mathbb{R}$ (Fig. 3). From the decomposition theorem and Proposition 6, we have

$$d(\tilde{a}, \tilde{b})(x) = \begin{cases} \frac{1}{2}x - \frac{1}{2} & \text{if } x \in [1,2], \\ \frac{1}{4}\sqrt{2x^2 - 4} & \text{if } x \in ]2, \sqrt{10}], \\ -\frac{1}{4}\sqrt{2x^2 - 4} + 2 & \text{if } x \in ]\sqrt{10}, \sqrt{34}], \\ 0 & \text{otherwise} \end{cases}$$

for each $x \in \mathbb{R}$ (Fig. 4).

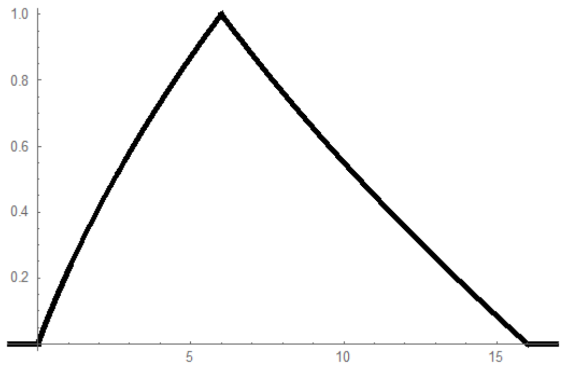


***Figure 3.*** $\langle \tilde{a}, \tilde{b} \rangle$.

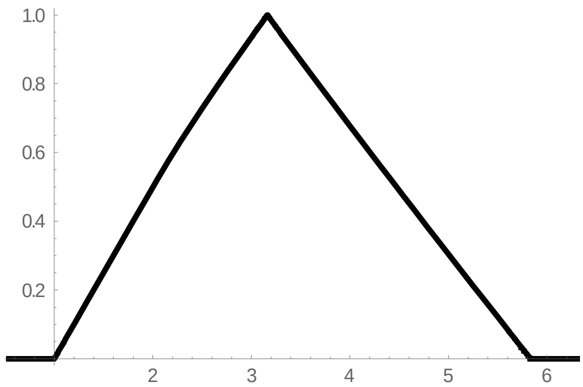


***Figure 4.*** $d(\tilde{a}, \tilde{b})$.

# 6. Conclusions

In the present paper, the results in [4] were extended to more general ones, and some useful results for applications were derived by the extended ones. Propositions 3.1 and 3.2 in [4] were extended to Propositions 2 and 3, respectively. Propositions 3.1 and 3.2 in [4] show that the class of images of level sets of one or two fuzzy sets under a mapping is the generator of another fuzzy set obtained from the one or two fuzzy sets by Zadeh's extension principle. Propositions 2 and 3 show that the class of images of generators of one or two fuzzy sets under a mapping is the generator of another fuzzy set obtained from the one or two fuzzy sets by Zadeh's extension principle. Then, Proposition 4 was derived from Proposition 3 in order to be more useful. By applying these results to fuzzy inner products, fuzzy distances, and operations of fuzzy sets, Propositions 5, 6, and 7 were derived. Propositions 5, 6, and 7 are very useful for analyzing fuzzy inner products, fuzzy distances, and operations of fuzzy sets. Furthermore, we discussed the application of the obtained results to the fuzzy minisum location problem. The obtained results in the present paper can be expected to be useful for analyzing various fuzzy mathematical programming problems such as the fuzzy minisum location problem.

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